COMPUTATION OF WEIGHT LATTICES OF G-VARIETIES

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ABSTRACT. Let G be a connected reductive group. To any irreducible G-variety X one assigns the lattice generated by all weights of B-semiinvariant rational functions on X, where B is a Borel subgroup of G. This lattice is called the weight lattice of X. We establish algorithms for computing weight lattices for homogeneous spaces and affine homogeneous vector bundles. For affine homogeneous spaces of rank $\operatorname{rk} G$ we present a more or less explicit computation.

CONTENTS

1. INTRODUCTION

Throughout the paper the base field is $\mathbb{C}$.

Let G be a connected reductive group and X be a normal irreducible G-variety. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$.

Consider the action of G on the field of rational functions $\mathbb{C}(X)$. The set of B-eigencharacters of this action form a subgroup in the character lattice $\mathfrak{X}(B)$ of B. This subgroup is

called the *weight lattice* of X and denoted by $\mathfrak{X}_{G,X}$. In other words,

$$\mathfrak{X}_{G,X} = \{\lambda \in \mathfrak{X}(B) | \exists f_\lambda \in \mathbb{C}(X), b.f_\lambda = \lambda(b)f_\lambda\}.$$

The weight lattice is an important invariant of a X. It is used, for instance, in the equivariant embedding theory of Luna and Vust, [LV]. By the rank of X (denoted by $\mathrm{rk}_G(X)$) we mean the rank of $\mathfrak{X}_{G,X}$.

The goal of this paper is compute or, rather, develop an algorithm computing the lattice $\mathfrak{X}_{G,X}$ for some classes of G-varieties. The classes in interest are homogeneous spaces G/H (H is an algebraic subgroup of G) and affine homogeneous vector bundle $G *_H V$ (H is a reductive subgroup of G and V is an H-module).

The basic result in the computation of the weight lattices for G-varieties is the reduction procedure due to Panyushev, [Pa1]. It reduces the computation of the weight lattice for X to that for an *affine* homogeneous space (in fact, together with some auxiliary datum – a point from a so-called distinguished component). For affine homogeneous spaces X the spaces $\mathfrak{a}_{G,X} := \mathfrak{X}_{G,X} \otimes_{\mathbb{Z}} \mathbb{C}$ were computed in [Lo4]. Moreover, in [Lo6] the author found a way to reduce the computation of $\mathfrak{X}_{G,X}$ to the case when $\mathrm{rk}_G(X) = \mathrm{rk}(G)$. We would like to note that this reduction traces back to another paper of Panyushev, [Pa2].

So, essentially, the only original result of this paper is the computation of $\mathfrak{X}_{G,X}$ for affine homogeneous spaces $X = G/H$ with $\mathrm{rk}_G(X) = \mathrm{rk}(G)$. The main result here is very technical Theorem 5.1.3.

The most important step in the computation of $\mathfrak{X}_{G,G/H}$ is computing a smaller lattice, the *root lattice* $\Lambda_{G,G/H}$ established by Knop, [K4]. In a sense, $\Lambda_{G,G/H}$ is an "essential" part of $\mathfrak{X}_{G,G/H}$. The advantage of $\Lambda_{G,G/H}$ over $\mathfrak{X}_{G,G/H}$ is a much better behavior. For example, $\Lambda_{G,G/H}$ depends only on the Lie algebra $\mathfrak{h}$ of H. Further, it is generated by a certain root system in $\mathfrak{a}_{G,G/H}$. The Weyl group of that root system is the so called *Weyl group $W_{G,G/H}$* of G/H. All groups $W_{G,G/H}$ were computed in author's preprint [Lo5]. The main result of the computation of $\Lambda_{G,G/H}$, Theorem 5.1.2 is much less technical than Theorem 5.1.3.

As in the computation of Weyl groups in [Lo5], the main ingredient in the computation of the lattices $\Lambda_{G,G/H}$ is the theory of Hamiltonian actions of reductive groups developed in [K1],[Lo1],[Lo2],[Lo6],[Lo7].

Let us describe briefly the content of the paper. In Section 2 we introduce conventions and the list of notation used in the paper. In Section 3 we review some known results and constructions related to weight and root lattices, including reductions of [Pa1],[Lo6] discussed above. Section 4 is devoted to the study of the root lattices of Hamiltonian actions. In Section 5 we state and prove our main results, Theorems 5.1.2,5.1.3. Finally, in Section 6 we briefly quote an algorithm for computing the weight lattices of homogeneous spaces and affine homogeneous vector bundles. Each of Sections 3-5 is divided into subsections, the first subsection of each of these sections describes its content in more detail.

2. Notation and conventions

For an algebraic group denoted by a capital Latin letter we denote its Lie algebra by the corresponding small German letter. For example, the Lie algebra of $\widetilde{L}_0$ is denoted by $\widetilde{\mathfrak{l}}_0$.

H-morphisms, H-subvarieties, etc. Let H be an algebraic group. We say that a variety X is an H-variety if an action of H on X is given. By an H-subset (resp., subvariety) in a given H-variety we mean an H-stable subset (resp., subvariety). A morphism of H-varieties is said to be an H-morphism if it is H-equivariant.

Borel subgroups and maximal tori. While considering a reductive group G, we always fix its Borel subgroup B and a maximal torus $T \subset B$. In accordance with this choice, we fix the root system $\Delta(\mathfrak{g})$ and the system of simple roots $\Pi(\mathfrak{g})$ of $\mathfrak{g}$. Let U denote the unipotent radical of B so that $B = T \ltimes U$.

If G_1, G_2 are reductive groups with fixed Borel subgroups $B_i \subset G_i$ and maximal tori $T_i \subset B_i$, then we take $B_1 \times B_2, T_1 \times T_2$ for the fixed Borel subgroup and maximal torus in $G_1 \times G_2$.

Suppose G_1 is a reductive algebraic group. Fix an embedding $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_1)$. Then $\mathfrak{t} \cap \mathfrak{g}_1$ is a Cartan subalgebra and $\mathfrak{b} \cap \mathfrak{g}_1$ is a Borel subalgebra of $\mathfrak{g}_1$. For fixed Borel subgroup and maximal torus in G_1 we take those with the Lie algebras $\mathfrak{b}_1, \mathfrak{t}_1$.

Homomorphisms and representations. All homomorphisms of reductive algebraic Lie algebras (for instance, representations) are assumed to be differentials of homomorphisms of the corresponding algebraic groups.

Identification $\mathfrak{g} \cong \mathfrak{g}^*$. Let G be a reductive algebraic group. There is a G-invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ such that its restriction to $\mathfrak{t}(\mathbb{R})$ is positively definite. For instance, if V is a locally effective G-module, then $(\xi, \eta) = \mathrm{tr}_V(\xi\eta)$ has the required properties. Note that if H is a reductive subgroup of G, then the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}$ is nondegenerate, so one may identify $\mathfrak{h}$ with $\mathfrak{h}^*$.

Parabolic subgroups and Levi subgroups. A parabolic subgroup of G is called *antistandard* if it contains the Borel subgroup B^- that contains T and is opposite to B. It is known that any parabolic subgroup is G-conjugate to a unique antistandard one. Antistandard parabolics are in one-to-one correspondence with subsets of $\Pi(\mathfrak{g})$. Namely, one assigns to $\Sigma \subset \Pi(\mathfrak{g})$ the antistandard parabolic subgroup, whose Lie algebra is generated by by $\mathfrak{b}^-$ and $\mathfrak{g}^\alpha, \alpha \in \Sigma$.

By a standard Levi subgroup in G we mean the Levi subgroup containing T of an antistandard parabolic subgroup.

Simple Lie algebras, their roots and weights. Simple roots of a simple Lie algebra $\mathfrak{g}$ are denoted by α_i. The numeration is described below. By π_i we denote the fundamental weight corresponding to α_i.

Classical algebras. In all cases for $\mathfrak{b}$ (resp. $\mathfrak{t}$) we take the algebra of all upper triangular (resp., diagonal) matrices in $\mathfrak{g}$.

$\mathfrak{g} = \mathfrak{sl}_n$. Let $e_1, \ldots, e_n$ denote the standard basis in $\mathbb{C}^n$ and $e^1, \ldots, e^n$ the dual basis in $\mathbb{C}^{n*}$. Choose the generators $\varepsilon_i, i = \overline{1, n}$, of $\mathfrak{t}^*$ given by $\langle \varepsilon_i, \mathrm{diag}(x_1, \ldots, x_n) \rangle = x_i$. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = \overline{1, n-1}$.

$\mathfrak{g} = \mathfrak{so}_{2n+1}$. Let $e_1, \ldots, e_{2n+1}$ be the standard basis in $\mathbb{C}^{2n+1}$. We suppose $\mathfrak{g}$ annihilates the form $(x, y) = \sum_{i=1}^{2n+1} x_i y_{2n+2-i}$. Define $\varepsilon_i \in \mathfrak{t}^*, i = \overline{1, n}$, by $\langle \varepsilon_i, \mathrm{diag}(x_1, \ldots, x_n, 0, -x_n, \ldots, -x_1) \rangle = x_i$. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = \overline{1, n-1}, \alpha_n = \varepsilon_n$.

$\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $e_1, \ldots, e_{2n}$ be the standard basis in $\mathbb{C}^{2n}$. We suppose that $\mathfrak{g}$ annihilates the form $(x, y) = \sum_{i=1}^{n} (x_i y_{2n+1-i} - y_i x_{2n+1-i})$. Let us define $\varepsilon_i \in \mathfrak{t}^*, i = \overline{1, n}$, by $\langle \varepsilon_i, \mathrm{diag}(x_1, \ldots, x_n, -x_n, \ldots, -x_1) \rangle = x_i$. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = \overline{1, n-1}, \alpha_n = 2\varepsilon_n$.

$\mathfrak{g} = \mathfrak{so}_{2n}$. Let $e_1, \ldots, e_{2n}$ be the standard basis in $\mathbb{C}^{2n}$. We suppose that $\mathfrak{g}$ annihilates the form $(x, y) = \sum_{i=1}^{2n} x_i y_{2n+1-i}$. Define $\varepsilon_i \in \mathfrak{t}^*, i = \overline{1, n}$, in the same way as for $\mathfrak{g} = \mathfrak{sp}_{2n}$. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = \overline{1, n-1}, \alpha_n = \varepsilon_{n-1} + \varepsilon_n$.

Exceptional algebras. For roots and weights of exceptional Lie algebras we use the notation from [OV]. The numeration of simple roots is also taken from [OV].

Subalgebras in semisimple Lie algebra. For semisimple subalgebras of exceptional Lie algebras we use the notation from [D]. Below we explain the notation for classical algebras.

Suppose $\mathfrak{g} = \mathfrak{sl}_n$. By $\mathfrak{sl}_k, \mathfrak{so}_k, \mathfrak{sp}_k$ we denote the subalgebras of $\mathfrak{sl}_n$ annihilating a subspace $U \subset \mathbb{C}^n$ of dimension $n-k$, leaving its complement V invariant, and (for $\mathfrak{so}_k, \mathfrak{sp}_k$) annihilating a nondegenerate orthogonal or symplectic form on V.

The subalgebras $\mathfrak{so}_k \subset \mathfrak{so}_n, \mathfrak{sp}_k \subset \mathfrak{sp}_n$ are defined analogously. The subalgebra $\mathfrak{gl}_k^{diag}$ is embedded into $\mathfrak{so}_n, \mathfrak{sp}_n$ via the direct sum of τ, τ^* and a trivial representation (here τ denotes the tautological representation of $\mathfrak{gl}_k$). The subalgebras $\mathfrak{sl}_k^{diag}, \mathfrak{so}_k^{diag}, \mathfrak{sp}_k^{diag} \subset \mathfrak{so}_n, \mathfrak{sp}_n$ are defined analogously. The subalgebra G_2 (resp., $\mathfrak{spin}_7$) in $\mathfrak{so}_n$ is the image of G_2 (resp., $\mathfrak{so}_7$) under the direct sum of the 7-dimensional irreducible (resp., spinor) and the trivial representations.

Finally, let $\mathfrak{h}_1, \mathfrak{h}_2$ be subalgebras of $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n$ described above. While writing $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, we always mean that $(\mathbb{C}^n)^{\mathfrak{h}_1} + (\mathbb{C}^n)^{\mathfrak{h}_2} = \mathbb{C}^n$.

The description above determines a subalgebra uniquely up to conjugacy in $\mathrm{Aut}(\mathfrak{g})$.

Now we list some notation used in the text.

$\sim_G$	the equivalence relation induced by an action of group G.
$A^{(B)}$	the subset of all B-semiinvariant functions in a G-algebra A.
$A^\times$	the group of all invertible elements of an algebra A.
$\mathrm{Aut}(\mathfrak{g})$	the group of automorphisms of a Lie algebra $\mathfrak{g}$.
$\mathrm{Aut}^G(X)$	the group of G-automorphisms of a G-variety X.
e_α	a nonzero element of the root subspace $\mathfrak{g}^\alpha$.
(G, G)	the commutant of a group G.
$[\mathfrak{g}, \mathfrak{g}]$	the commutant of a Lie algebra $\mathfrak{g}$.
G°	the connected component of unit of an algebraic group G.
$G *_H V$	the homogeneous bundle over G/H with fiber V.
$[g, v]$	the equivalence class of (g, v) in $G *_H V$.
G_x	the stabilizer of $x \in X$ under an action $G : X$.
$\mathfrak{g}^\alpha$	the root subspace of $\mathfrak{g}$ corresponding to a root α.
$\mathfrak{g}^{(A)}$	the subalgebra $\mathfrak{g}$ generated by $\mathfrak{g}^\alpha$ with $\alpha \in A \cup -A$.
$G^{(A)}$	the connected subgroup of G with Lie algebra $\mathfrak{g}^{(A)}$.
$m_G(X)$	$:= \max_{x \in X} \dim Gx$.
$N_G(H)$ $(N_G(\mathfrak{h}))$	the normalizer of a subgroup H (subalgebra $\mathfrak{h} \subset \mathfrak{g}$) in a group G.
$\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$	the normalizer of a subalgebra $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$.
$\mathrm{rk}(G)$	the rank of an algebraic group G.
$\mathrm{R}_u(H)$ $(\mathrm{R}_u(\mathfrak{h}))$	the unipotent radical of an algebraic group H (of an algebraic Lie algebra $\mathfrak{h}$).
s_α	the reflection in a Euclidian space corresponding to a vector α.
$V^\mathfrak{g}$	$= \{v \in V \| \mathfrak{g}v = 0\}$, where $\mathfrak{g}$ is a Lie algebra and V is a $\mathfrak{g}$-module.
$V(\lambda)$	the irreducible module of the highest weight λ over a reductive algebraic group or a reductive Lie algebra.
$W(\mathfrak{g})$	the Weyl group of a reductive Lie algebra $\mathfrak{g}$.
$\mathfrak{X}(G)$	the character lattice of an algebraic group G.
$\mathfrak{X}_G$	the weight lattice of a reductive algebraic group G.
X^G	the fixed point set for an action $G : X$.
$X//G$	the categorical quotient for an action $G : X$, where G is a reductive group and X is an affine G-variety.
$\#X$	the number of elements in a set X.

$Z_G(H), (Z_G(\mathfrak{h}))$	the centralizer of a subgroup H (of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$) in an algebraic group G.
$Z(G)$	$:= Z_G(G)$.
$\mathfrak{z}_\mathfrak{g}(\mathfrak{h})$	the centralizer of a subalgebra $\mathfrak{h}$ in $\mathfrak{g}$.
$\mathfrak{z}(\mathfrak{g})$	$:= \mathfrak{z}_\mathfrak{g}(\mathfrak{g})$.
$\alpha^\vee$	the dual root to α.
$\Delta(\mathfrak{g})$	the root system of a reductive Lie algebra $\mathfrak{g}$.
λ^*	the dual highest weight to λ.
$\Lambda(\mathfrak{g})$	the root lattice of a reductive Lie algebra $\mathfrak{g}$.
$\Lambda_{G,X}$	the root lattice of a G-variety X.
ξ_s, ξ_n	semisimple and nilpotent parts of an element ξ in an algebraic Lie algebra.
$\Pi(\mathfrak{g})$	the system of simple roots for a reductive Lie algebra $\mathfrak{g}$.
$\pi_{G,X}$	the (categorical) quotient morphism $X \to X/\!/G$.

3. Known results and constructions

3.1. **Introduction.** This section does not contain new results. In the first subsection we quote definitions and basic properties of central automorphisms and root lattices of G-varieties. Our exposition is based mostly on [K4]. In the second subsection we show how to reduce the computation of $\mathfrak{X}_{G,X}$ to the case when X is an affine homogeneous space of rank $\mathrm{rk}\, G$. The reduction is based on results of [Pa1],[Lo6].

3.2. **Central automorphisms and root lattices.** Let G, X be such as in Introduction. The following definition was given in [K4].

Definition 3.2.1. A G-automorphism φ of X is said to be *central* if for any $\lambda \in \mathfrak{X}_{G,X}$ the automorphism φ acts on $\mathbb{C}(X)_\lambda^{(B)}$ by a constant. Central automorphisms of X form a group denoted by $\mathfrak{A}_G(X)$.

However, the group $\mathfrak{A}_G(X)$ has a disadvantage not to be a birational invariant of X. This problem is fixed as follows. By [K4], Theorem 5.1, any open G-subvariety $X^0 \subset X$ is stable with respect to $\mathfrak{A}_G(X)$ and there is the inclusion $\mathfrak{A}_G(X) \subset \mathfrak{A}_G(X^0)$. It turns out that there is a unique maximal group of the form $\mathfrak{A}_G(X^0)$ ([K4], Corollary 5.4). We denote this group by $\mathfrak{A}_{G,X}$.

Lemma 3.2.2 ([Lo6], Lemma 7.17). *If X is quasiaffine, then $\mathfrak{A}_{G,X} = \mathfrak{A}_G(X)$.*

Set $A_{G,X} := \mathrm{Hom}(\mathfrak{X}_{G,X}, \mathbb{C}^\times)$. The group $\mathfrak{A}_{G,X}$ is embedded into $A_{G,X}$ as follows. We assign $a_{\varphi,\lambda} \in \mathbb{C}^\times$ to $\varphi \in \mathfrak{A}_{G,X}, \lambda \in \mathfrak{X}_{G,X}$ by $\varphi f_\lambda = a_{\varphi,\lambda} f_\lambda$, $f \in \mathbb{C}(X)_\lambda^{(B)}$. The map $\iota_{G,X} : \mathfrak{A}_{G,X} \to A_{G,X}$ is defined by $\lambda(\iota_{G,X}(\varphi)) = a_{\varphi,\lambda}$. Clearly, $\iota_{G,X}$ is a well-defined group homomorphism.

Proposition 3.2.3 ([K4], Theorem 5.5). *The map $\iota_{G,X}$ is injective and its image is closed.*

The following lemma justifies the term "central":

Lemma 3.2.4 ([K4], Corollary 5.6). $\mathfrak{A}_G(X)$ *is contained in the center of* $\mathrm{Aut}^G(X)$.

Definition 3.2.5. The lattice $\Lambda_{G,X} \subset \mathfrak{X}_{G,X} = \mathfrak{X}(A_{G,X})$ consisting of all characters annihilating $\iota_{G,X}(\mathfrak{A}_{G,X})$ is called the *root lattice* of X.

Since the image of $Z(G)$ in $\mathrm{Aut}^G(X)$ lies in $\mathfrak{A}_{G,X}$, we see that $\Lambda_{G,X} \subset \Lambda(\mathfrak{g})$.

Proposition 3.2.6 ([K4], Theorem 6.3). *Let X_1, X_2 be irreducible G-varieties and $\varphi : X_1 \to X_2$ a dominant generically finite G-morphism. Then $\Lambda_{G,X_1} = \Lambda_{G,X_2}$.*

In particular, the lattice $\Lambda_{G,G/H}$ depends only on the pair $\mathfrak{g}, \mathfrak{h}$, so we write $\Lambda(\mathfrak{g}, \mathfrak{h})$ instead of $\Lambda_{G,G/H}$.
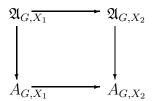
Proposition 3.2.7. $\Lambda_{G,X} = \Lambda_{G,Gx}$ *for $x \in X$ in general position.*

Proof. This follows directly from [K4], Theorem 5.9. □

Lemma 3.2.8. *Let X_1, X_2 be homogeneous G-spaces and $\varphi : X_1 \to X_2$ a dominant G-morphism. Then $\mathfrak{X}_{G,X_2} \subset \mathfrak{X}_{G,X_1}$. Suppose X_1, X_2 are, in addition, quasiaffine. Then $\Lambda_{G,X_2} \subset \Lambda_{G,X_1}$ and there exists a unique homomorphism $\mathfrak{A}_{G,X_1} \to \mathrm{Aut}^G(X_2)$ such that φ becomes $\mathfrak{A}_{G,X_1}$-equivariant. Its image is contained in $\mathfrak{A}_{G,X_2}$. The dual to the corresponding homomorphism $\mathfrak{A}_{G,X_1} \to \mathfrak{A}_{G,X_2}$ coincides with the homomorphism $\mathfrak{X}_{G,X_2}/\Lambda_{G,X_2} \to \mathfrak{X}_{G,X_1}/\Lambda_{G,X_1}$ induced by the inclusions of lattices.*

Proof. The claim on inclusions of the weight lattices is clear. Below X_1, X_2 are quasiaffine.

Recall that $\mathfrak{A}_{G,X_1}$ acts on X_1 by central G-automorphisms (Lemma 3.2.2). The subalgebra $\mathbb{C}[X_2] \subset \mathbb{C}[X_1]$ is G-stable whence $\mathfrak{A}_{G,X_1}$-stable. From the existence of a T-embedding $\mathbb{C}(X_2)^{(B)} \hookrightarrow \mathbb{C}(X_1)^{(B)}$ it follows that $\mathfrak{A}_{G,X_1}$ acts on X_2 by central automorphisms. This observation implies that the required homomorphism $\mathfrak{A}_{G,X_1} \to \mathrm{Aut}^G(X_2)$ exists and is unique.

Consider the homomorphism $\mathfrak{A}_{G,X_1} \to \mathfrak{A}_{G,X_2}$ we have just constructed and the natural epimorphism $A_{G,X_1} \twoheadrightarrow A_{G,X_2}$. It is clear that the following diagram is commutative.

$$
\begin{array}{ccc}
\mathfrak{A}_{G,\overline{X_1}} & \longrightarrow & \mathfrak{A}_{G,X_2} \\
\downarrow & & \downarrow \\
A_{G,\overline{X_1}} & \longrightarrow & A_{G,X_2}
\end{array}
$$

Therefore all elements of $\Lambda_{G,X_2} \subset \mathfrak{X}_{G,X_2} \subset \mathfrak{X}_{G,X_1}$ are annihilated on $\mathfrak{A}_{G,X_1}$ whence the inclusion of root lattices. The homomorphism $\mathfrak{A}_{G,X_1} \to \mathfrak{A}_{G,X_2}$ possesses the required properties. □

Now we present the definition of the root system of a G-variety. To this end we need the following general construction.

Definition 3.2.9. Let V be a finitely dimensional euclidian vector space, Λ a lattice in V, Γ a finite subgroup in $\varnothing(V)$ generated by reflections and stabilizing Λ. By the *minimal root system* associated with Γ, Λ we mean the set consisting of all primitive $v \in \Lambda$ such that $s_v \in \Gamma$.

It is easy to see that any minimal root system is a genuine reduced root system.

Recall that to X one assigns the finite group $W_{G,X} \in \mathrm{GL}(\mathfrak{a}_{G,X})$ generated by reflections and stabilizing $\mathfrak{X}_{G,X}$ (see, for example, [Lo5], Theorem 1.1.4). By [K4], Corollary 6.2, $\Lambda_{G,X}$ is $W_{G,X}$-stable.

Definition 3.2.10. The *root system* $\Delta_{G,X}$ of X is the minimal root system associated with $W_{G,X}, \Lambda_{G,X}$.

The root system of X has some properties analogous to those of the root system of a reductive Lie algebra.

Proposition 3.2.11 ([K4], Corollary 6.5). *$\Delta_{G,X}$ generates $\Lambda_{G,X}$ and the Weyl group of $\Delta_{G,X}$ coincides with $W_{G,X}$.*

Now let us state one results regarding $\Lambda_{G,X}$ obtained in [Lo6].

Proposition 3.2.12 ([Lo6], Proposition 8.8). *Suppose that X is quasiaffine and $\mathrm{rk}_G(X) = \mathrm{rk}\, G$. Let $G = Z(G)^\circ G_1 \ldots G_k$ be the decomposition of G into the locally direct product of the unit component of the center and simple normal subgroups. Then $\Lambda_{G,X} = \bigoplus_{i=1}^k \Lambda_{G_i,X}$.*

3.3. **Reduction of the computation of $\mathfrak{X}_{G,X}$.** At first, let us explain the reduction of computing $\mathfrak{X}_{G,X}$ for homogeneous spaces X to that for affine homogeneous vector bundles.

Let H be an algebraic subgroup of G. It is known that there a parabolic subgroup $Q \subset G$ and its Levi subgroup M such that $\mathrm{R}_u(H) \subset \mathrm{R}_u(Q)$ and $S := M \cap H$ is a maximal reductive subgroup of H. Replacing H with a conjugate subgroup, we may assume that Q is antistandard and M is standard.

The following result is due to Panyushev, [Pa1], cf. [Lo5], Proposition 3.2.9.

Proposition 3.3.1. *Let Q, M, S, H be as above. Then $\mathfrak{X}_{G,G/H} = \mathfrak{X}_{M,M*_S(\mathrm{R}_u(\mathfrak{q})/\,\mathrm{R}_u(\mathfrak{h}))}$.*

Next, we are going to reduce the computation for affine homogeneous vector bundles to that for affine homogeneous spaces.

First of all, set

$$(3.1) \qquad\qquad L_{G,X} := Z_G(\mathfrak{a}_{G,X}).$$

$$(3.2) \qquad\qquad L_{0\,G,X} := \{g \in L | \chi(g) = 1, \forall \chi \in \mathfrak{X}_{G,X}\}.$$

Definition 3.3.2. Let X be a smooth quasiaffine G-variety, L_1 a normal subgroup of $L_{0\,G,X}$. There is a unique irreducible (=connected) component $\underline{X} \subset X^{L_1}$ such that $\overline{U\underline{X}} = X$ (see [Lo6], Proposition 8.4). This component $\underline{X} \subset X^{L_1}$ is said to be *distinguished*.

In the sequel we will need to extend the definition of $L_{0\,G,X}$ to actions of certain disconnected groups G.

Definition 3.3.3. A reductive algebraic group $\widetilde{G}$ is called *almost connected* if $\widetilde{G} = \widetilde{G}^\circ Z(\widetilde{G})$.

Let $\widetilde{G}$ be an almost connected group with $\widetilde{G}^\circ = G$. Set $\widetilde{B} := N_{\widetilde{G}}(B), \widetilde{T} := Z_{\widetilde{G}}(T)$. Since the group $\widetilde{G}$ is almost connected, we see that $\widetilde{T} \subset \widetilde{B}$ and $\widetilde{B} = \widetilde{T} \ltimes U$. So the groups $\mathfrak{X}(\widetilde{B})$ and $\mathfrak{X}(\widetilde{T})$ are identified.

For an irreducible $\widetilde{G}$-variety X put

$$(3.3) \qquad\qquad \mathfrak{X}_{\widetilde{G},X} := \{\chi \in \mathfrak{X}(\widetilde{B}) | \exists f \in \mathbb{C}(X) | b.f = \chi(b)f, \forall b \in \widetilde{B}\}.$$

By definition, put $\mathfrak{a}_{\widetilde{G},X} = \mathfrak{a}_{G,X}$. The subgroups $L_{\widetilde{G},X}, L_{0\,\widetilde{G},X} \subset \widetilde{G}$ are defined by formulas (3.1),(3.2), resp., where G is replaced with $\widetilde{G}$. It follows directly from definition that $L_{0\,\widetilde{G},X}$ is almost connected.

The following proposition is essentially due to Panyushev, [Pa1], and is proved in the same way as its analogue in [Lo5], Proposition 3.2.12.

Proposition 3.3.4. *Let H be a reductive subgroup in G, V an H-module and π the natural projection $G *_H V \to G/H$. Put $L_1 = L_{0\,G,G/H}$. Let x be a point from the distinguished component of $(G/H)^{L_1}$. Then $L_{0\,G,G*_H V} = L_{0\,L_1,\pi^{-1}(x)}$.*

The last proposition reduces the computation of $\mathfrak{X}_{G,G*_H V}$ to the following problems:

(1) To determine the lattice $\mathfrak{X}_{G,G/H}$, equivalently, the group $L_{0\,G,G/H}$ for all reductive subgroups $H \subset G$.
(2) To find a point from the distinguished component of $(G/H)^{L_0\,G,G/H}$ for all reductive subgroups $H \subset G$.
(3) To find the group $L_{0\,\widetilde{G},V}$ for almost connected group $\widetilde{G}$ and a $\widetilde{G}$-module V.

There is an algorithm solving the third problem. It is presented, for instance, in [Pa4] for connected $\widetilde{G}$. This algorithm can be generalized directly to the general case. We quote this algorithm in Section 6.

Next, we reduce the computation of $\mathfrak{X}_{G,X}$ and the determination of a point in the distinguished component to the case, where X is an affine homogeneous space such that $\mathrm{rk}_G(G/H) = \mathrm{rk}(G)$.

Proposition 3.3.5. *Let X be a smooth quasiaffine G-variety, $L_0 := L_{0\,G,X}$, and $\underline{X}$ the distinguished component of $X^{L_0^\circ}$. Set $\underline{G} = N_G(L_0^\circ, \underline{X})/L_0^\circ$. Then $\mathfrak{X}_{G,X} = \mathfrak{X}_{\underline{G}^\circ, \underline{X}}$ and the distinguished components of X^{L_0} and $\underline{X}^{L_0/L_0^\circ}$ coincide. Here $\underline{X}$ is considered as a $\underline{G}$-variety.*

Proof. The equality of the weight lattices was proved in [Lo6], Theorem 8.7. Further, $\underline{X}^{L_0/L_0^\circ}$ is a union of components of X^{L_0}. Let $\underline{X}'$ be the distinguished component of $\underline{X}^{L_0/L_0^\circ}$. Then $(U \cap N_G(L_0^\circ, \underline{X}))\underline{X}'$ is dense in $\underline{X}$. It follows that $U\underline{X}'$ is dense in X whence $\underline{X}'$ is the distinguished component of X^{L_0}. $\qquad\square$

Now let $X = G/H$ be an affine homogeneous space. The Cartan spaces $\mathfrak{a}_{G,X}$ were computed in [Lo4], distinguished components $\underline{X}$ were determined in [Lo5], Section 4. It turns out that $\underline{X}$ is an affine homogeneous $\underline{G}^\circ$-space. So one can reduce the computation of $\mathfrak{X}_{G,X}$ to the case when X is an affine homogeneous space of rank $\mathrm{rk}(G)$.

4. Root lattices of Hamiltonian actions

4.1. **Introduction.** In the first subsection we define Hamiltonian actions in the algebraic context, give some examples, state the symplectic slice theorem that provides a local description of an affine Hamiltonian varieties. Finally, we define an important special class of affine Hamiltonian varieties: conical varieties.

In the second subsection we recall some definitions and results related to Weyl groups, weight and root lattices of Hamiltonian varieties. For simplicity, we consider only Hamiltonian G-varieties X such that $m_G(X) = \dim G$. After giving all necessary definitions we state the comparison theorem (Theorem 4.3.2) that relates the Weyl group, the weight and root lattices of an affine G-variety X_0 with those of $X = T^*X_0$. Finally, we state several technical results to be used in Subsection 4.4. The most important among them are Propositions 4.3.4,4.3.5. Subsections 4.2,4.3 do not contain new results.

In Subsection 4.4 we study root lattices of affine Hamiltonian G-varieties from a certain class. This class includes all cotangent bundles T^*X_0, where X_0 is an affine G-variety such that $\mathrm{rk}_G(X_0) = \mathrm{rk}\,G$. These results (Propositions 4.4.6,4.4.9, Corollary 4.4.8) play a crucial role in the proof of Theorem 5.1.2.

4.2. **Preliminaries.** Let X be a symplectic variety with symplectic form ω and G a reductive algebraic group acting on X by symplectomorphisms. This action is called *Hamiltonian* if it is equipped with a linear G-equivariant map $\mathfrak{g} \to \mathbb{C}[X], \xi \mapsto H_\xi$, such that $\{H_\xi, f\} = \xi_* f$. Here ξ_* denotes the velocity vector field associated with ξ. The variety X is called a *Hamiltonian G-variety*. The *moment map* of X is the morphism $X \to \mathfrak{g}^*$ defined

by $\langle \mu_{G,X}(x), \xi \rangle = H_\xi(x)$. In the sequel we fix a G-invariant nondegenerate symmetric form $(\cdot, \cdot)$ on $\mathfrak{g}$ and identify $\mathfrak{g}$ with $\mathfrak{g}^*$.

We say that an irreducible Hamiltonian G-variety X is *coisotropic* if a G-orbit of X in general position is a coisotropic G-variety. If $m_G(X) = \dim G$, then X is coisotropic iff $\dim X = \dim G + \operatorname{rk} G$, see, for instance, [Lo7], Lemma 2.22.

Let us present three examples of Hamiltonian G-varieties.

Example 4.2.1 (Symplectic vector spaces). Let V be a symplectic vector space and G be a reductive group acting on V by linear symplectomorphisms. Then the action $G : V$ is Hamiltonian. The moment map $\mu_{G,V}$ is given by $\langle \mu_{G,V}(v), \xi \rangle = \frac{1}{2}\omega(\xi v, v), \xi \in \mathfrak{g}, v \in V$.

Example 4.2.2 (Cotangent bundles). Let Y be a smooth G-variety. Let X be the cotangent bundle of Y. Then X is a symplectic algebraic variety. The action of G on X is Hamiltonian. The moment map is given by $\langle \mu_{G,X}((y, \alpha)), \xi \rangle = \langle \alpha, \xi_* y \rangle$. Here $y \in Y, \alpha \in T_y^* Y, \xi \in \mathfrak{g}$.

Example 4.2.3 (Model varieties). This example was introduced in [Lo1]. It generalizes Example 4.2.1 and partially Example 4.2.2. Let H be a reductive subgroup of G, $\eta \in \mathfrak{g}^H$, V a symplectic H-module. Put $U = (\mathfrak{z}_{\mathfrak{g}}(\eta)/\mathfrak{h})^*$. There is a certain closed G-invariant 2-form ω on the homogeneous vector bundle $X = G *_H (U \oplus V)$ depending on the choice of an $\mathfrak{sl}_2$-triple (η_n, h, f) in $\mathfrak{z}_{\mathfrak{g}}(\eta_s)^H$, see [Lo1] or [Lo7], Example 2.5. By the model variety $M_G(H, \eta, V)$ we mean the set of all points of X, where ω is nondegenerate. It was proved in [Lo1] that $G/H \subset M_G(H, \eta, V)$ and $X = M_G(H, \eta, V)$ for nilpotent η. By the base point of $M_G(H, \eta, V)$ we mean $[1, (0, 0)] \in G *_H (U \oplus V)$. The moment map of $M_G(H, \eta, V)$ is constructed as follows. Identify U with $\mathfrak{z}_{\mathfrak{g}}(\eta_s + f) \cap \mathfrak{h}^\perp$ by means of $(\cdot, \cdot)$. Then

$$\mu_{G,M_G(H,\eta,V)}([g, (u, v)]) = \operatorname{Ad}(g)(\eta + u + \mu_{H,V}(v)).$$

Actually, the Hamiltonian structure on $M_G(H, \eta, V)$ does not depend on the choice of h, f up to an isomorphism.

If $\eta = 0, H = G$ (resp., $\eta = 0, V = \{0\}$), $M_G(H, \eta, V)$ is the symplectic vector space V (resp., the cotangent bundle $T^*(G/H)$).

Let us explain why the previous example is important. Let X be an affine Hamiltonian G-variety and x a point in X with closed G-orbit. It turns out that in a small neighborhood of x the variety X looks like a model variety.

To state a precise result we define some invariants of the triple (G, X, x). Put $H = G_x, \eta = \mu_{G,X}(x)$. The subgroup $H \subset G$ is reductive and $\eta \in \mathfrak{g}^H$. Put $V = (\mathfrak{g}_* x)^\angle/(\mathfrak{g}_* x \cap \mathfrak{g}_* x^\angle)$. This is a symplectic H-module. We say that (H, η, V) is the *determining triple* of X at x. For example, the determining triple of $X = M_G(H, \eta, V)$ in $x = [1, (0, 0)]$ is (H, η, V), see [Lo1], assertion 4 of Proposition 1.

Definition 4.2.4. Let X_1, X_2 be affine Hamiltonian G-varieties, $x_1 \in X_1, x_2 \in X_2$ be points with closed G-orbits. The pairs $(X_1, x_1), (X_2, x_2)$ are called *analytically equivalent*, if there are saturated open analytical neighborhoods O_1, O_2 of $x_1 \in X_1, x_2 \in X_2$, respectively, and an isomorphism $O_1 \to O_2$ of complex-analytical Hamiltonian G-manifolds that maps x_1 to x_2.

Recall that a subset of an affine G-variety X is said to be *saturated* if it is the union of fibers of $\pi_{G,X}$.

Remark 4.2.5. An open saturated analytical neighborhood in X is the inverse image of an *open* analytical neighborhood in $X/\!/G$ under $\pi_{G,X}$. See, for example, [Lo1], Lemma 5.

Proposition 4.2.6 (Symplectic slice theorem, [Lo1])**.** *Let X be an affine Hamiltonian G-variety, $x \in X$ a point with closed G-orbit, (H, η, V) the determining triple of X at x, and x' the base point of $M_G(H, \eta, V)$. Then the pair (X, x) is analytically equivalent to the pair $(M_G(H, \eta, V), x')$.*

In the sequel we will often consider Hamiltonian varieties equipped with an action of $\mathbb{C}^\times$ satisfying some compatibility conditions. Here is the precise definition.

Definition 4.2.7. An affine Hamiltonian G-variety X equipped with an action $\mathbb{C}^\times : X$ commuting with the action of G is said to be *conical* if the following conditions (Con1),(Con2) are fulfilled

(Con1) The morphism $\mathbb{C}^\times \times X/\!/G \to X/\!/G, (t, \pi_{G,X}(x)) \mapsto \pi_{G,X}(tx)$, can be extended to a morphism $\mathbb{C} \times X/\!/G \to X/\!/G$.
(Con2) There exists a positive integer k (called the *degree* of X) such that $t.\omega = t^k \omega$ and $\mu_{G,X}(tx) = t^k \mu_{G,X}(x)$ for all $t \in \mathbb{C}^\times, x \in X$.

Example 4.2.8 (Cotangent bundles)**.** Let Y, X be such as in Example 4.2.2. The variety X is a vector bundle over Y. The action $\mathbb{C}^\times : X$ by the fiberwise multiplication turns X into a conical variety of degree 1.

Example 4.2.9 (Model varieties)**.** Let H, η, V be such as in Example 4.2.3 and $X = M_G(H, \eta, V)$. Suppose that η is nilpotent. Here we define an action $\mathbb{C}^\times : X$ turning X into a conical Hamiltonian variety of degree 2. Let (η, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}^H$. Note that h is the image of a coroot under an embedding of Lie algebras. In particular, there exists a one-parameter subgroup $\gamma : \mathbb{C}^\times \to G$ with $\frac{d}{dt}|_{t=0}\gamma = h$. Since $[h, \mathfrak{h}] = 0, [h, f] = -2f$, we see that $\gamma(t)(\mathfrak{h}^\perp) = \mathfrak{h}^\perp, \gamma(t)(U) = U$. Define a morphism $\mathbb{C}^\times \times X \to X$ by formula

$$(4.1) \qquad (t, [g, (u, v)]) \mapsto [g\gamma(t), t^2 \gamma(t)^{-1}u, tv], t \in \mathbb{C}^\times, g \in G, u \in U, v \in V.$$

(Con1),(Con2) were checked in [Lo7], Example 2.16.

4.3. **Weyl groups and root lattices for Hamiltonian varieties.** In this section X is an irreducible affine Hamiltonian G-variety with symplectic form ω such that $m_G(X) = \dim G$. It is known that the last condition is equivalent to $\overline{\operatorname{im} \mu_{G,X}} = \mathfrak{g}$. For a Levi subalgebra $l \subset \mathfrak{g}$ set $l^{pr} := \{\xi \in l | \mathfrak{z}_{\mathfrak{g}}(\xi_s) \subset l\}$. Set $X^{pr} := G\mu_{G,X}^{-1}(\mathfrak{t}^{pr})$. By Propositions 4.1, 4.4 from [Lo6], the following claims take place:

   (1) the variety $Y := \mu_{G,X}^{-1}(l^{pr})$ is smooth,
   (2) the restriction of ω to Y is nondegenerate,
   (3) the action $L : Y$ is Hamiltonian with moment map $\mu_{G,X}|_Y$,
   (4) the natural morphism $G *_{N_G(L)} Y \to L$ is etale. Moreover, if $L = T$, then this morphism is an open embedding with image X^{pr}. In particular, the group $N_G(T)$ permutes transitively connected components of Y.

A component of $\mu_{G,X}^{-1}(l^{pr})$ is said to be an L-cross-section of X. Let us fix a T-cross-section X_T. By the Weyl group of the Hamiltonian variety X (associated with X_T) we mean the group $W_{G,X}^{(X_T)} := N_G(T, X_T)/T$ considered as a linear group acting on $\mathfrak{t}$.

Set $\psi_{G,X} := \pi_{G,\mathfrak{g}} \circ \mu_{G,X} : X \to \mathfrak{g}/\!/G$. It turns out, see [Lo2], Subsection 5.2, that there is a unique morphism $\widehat{\psi}_{G,X} : X \to \mathfrak{t}/\!/W_{G,X}^{(X_T)}$ such that $\psi_{G,X}$ is the composition of $\widehat{\psi}_{G,X}$ and the natural finite morphism $\mathfrak{t}/\!/W_{G,X}^{(X_T)} \to \mathfrak{g}/\!/G$.

Definition 4.3.1. Suppose X is conical. We say that X is *untwisted* if $W_{G,X}^{(X_T)}$ is generated by reflections and the morphism $\widehat{\psi}_{G,X}$ is smooth in codimension 1 (that is, the subvariety of singular points of $\widehat{\psi}_{G,X}$ has codimension at least 2 in X).

Proceed to the definitions of weight and root lattices of X. Let T_0 be the inefficiency kernel for the action $T : X_T$. Thanks to (4), T_0 is a discrete subgroup of T. By the *weight lattice* of X we mean the annihilator of T_0 in $\mathfrak{X}(T)$, we denote the weight lattice by $\mathfrak{X}_{G,X}^{(X_T)}$. Finally, let us define the root lattice of X. We say that a Hamiltonian (that is, G-equivariant and preserving ω and $\mu_{G,X}$) automorphism φ of X is *central* if $\varphi(X_T) = X_T$ and the restriction of φ to X_T coincides with the translation by some element $t_\varphi \in T/T_0$. Central automorphisms form a subgroup of $\mathrm{Aut}^G(X)$ denoted by $\mathfrak{A}_{G,X}^{(\cdot)}$. This subgroup does not depend on the choice of X_T. The map $\mathfrak{A}_{G,X}^{(\cdot)} \to T/T_0, \varphi \mapsto t_\varphi$ is injective, its image $\mathfrak{A}_{G,X}^{(X_T)}$ is closed, see [Lo6], Corollary 5.7. By the root lattice of X (denoted $\Lambda_{G,X}^{(X_T)}$) we mean the annihilator of $\mathfrak{A}_{G,X}^{(X_T)}$ in $\mathfrak{X}(T)$.

Theorem 4.3.2. *Let X_0 be an irreducible smooth affine G-variety of rank* $\mathrm{rk}\, G$. *Set* $X := T^*X_0$. *Then the following conditions hold:*
  (1) $m_G(X) = \dim G$.
  (2) X *is untwisted.*
  (3) *There is a T-cross-section Σ of X such that $W_{G,X}^{(\Sigma)} = W_{G,X_0}, \mathfrak{X}_{G,X}^{(\Sigma)} = \mathfrak{X}_{G,X_0}, \Lambda_{G,X}^{(\Sigma)} = \Lambda_{G,X_0}$.*
  (4) $L_{0\,G,X_0}$ *is the stabilizer in general position for the action $G : X$.*
  (5) X *is coisotropic iff X_0 is spherical, that is, B has an open orbit on X.*

Proof. (4) was proved in [K1], Korollar 8.2. Thence (1). (3) essentially was proved by Knop in [K3],[K4], see [Lo6], Theorem 7.8. Finally, (2) stems from the equality of the Weyl groups in (3) and [K5], Corollary 7.6 (see also [Lo7], Theorem 5.7, Remark 5.11). (5) was proved in [K1], Satz 7.1. $\qquad\square$

Lemma 4.3.3 ([Lo6], Lemma 6.12)**.** *The lattices $\Lambda_{G,X}^{(X_T)}, \mathfrak{X}_{G,X}^{(X_T)}$ are $W_{G,X}^{(X_T)}$-stable and $w\xi - \xi \in \Lambda_{G,X}^{(X_T)}$ for all $w \in W_{G,X}^{(X_T)}, \xi \in \mathfrak{X}_{G,X}^{(X_T)}$.*

The following proposition follows from [Lo7], Propositions 4.1,4.3.

Proposition 4.3.4. *Let X, T, X_T be such as above and M a Levi subgroup of G. Suppose $0 \in \mathrm{im}\, \widehat{\psi}_{G,X}$. Let $\xi \in \mathfrak{z}(\mathfrak{m})$ be a point in general position. Then there is a point $x \in X$ satisfying the following conditions*
  (a) $\mu_{G,X}(x)_s \in \mathfrak{z}(\mathfrak{m}) \cap \mathfrak{m}^{pr}$.
  (b) *A unique M-cross-section X_M of X containing x contains X_T and $\widehat{\psi}_{M,X_M}(x) = \pi_{W_{M,X_M}^{(X_T)},\mathfrak{t}}(\xi)$.*
  (c) Gx *is closed in X. Set $\widehat{G} := (M, M)$. Automatically, $\widehat{G}x$ is closed in X_M.*
  (d) *The Hamiltonian $\widehat{G}$-variety $\widehat{X} := M_{\widehat{G}}(H \cap \widehat{G}, \eta_n, V/V^H)$ is coisotropic, where (H, η, V) is the determining triple of X_M (or, equivalently, of X, see [Lo7], Lemma 2.27) at x.*
  (e) $H^\circ \subset \widehat{G}$.

It is easy to see that $m_{\widehat{G}}(X_M) = \dim \widehat{G}$. The slice theorem (Proposition 4.2.6) implies $m_{\widehat{G}}(\widehat{X}) = \dim \widehat{G}$. Thus the condition that $\widehat{X}$ is coisotropic means $\dim \widehat{X} = \dim \widehat{G} + \mathrm{rk}\, \widehat{G}$.

Proposition 4.3.5. *Let $X, T, X_T, M, X_M, \widehat{G}$ be such as in Proposition 4.3.4, $\widehat{T} := T \cap \widehat{G}$. Suppose $x \in X$ satisfies conditions (a)-(e) of Proposition 4.3.4. Let $\widehat{X}$ denote the model variety constructed from x in Proposition 4.3.4. Then there is a $\widehat{T}$-section $\widehat{X}_{\widehat{T}}$ of $\widehat{X}$ satisfying the following conditions.*

(1) *There is the inclusion*

$$(4.2) \qquad W_{\widehat{G}, \widehat{X}}^{(\widehat{X}_{\widehat{T}})} \subset (W_{G,X}^{(X_T)}) \cap M/T.$$

If X is conical and untwisted, then $\widehat{X}$ is also untwisted, and (4.2) turns into an equality.

(2) *Let p denote the orthogonal projection $\mathfrak{t} \to \widehat{\mathfrak{t}}$. Then*

$$(4.3) \qquad p(\mathfrak{X}_{G,X}^{(X_T)}) = \mathfrak{X}_{\widehat{G}, \widehat{X}}^{(\widehat{X}_{\widehat{T}})}.$$

(3) *There is the inclusion*

$$(4.4) \qquad \Lambda_{\widehat{G}, \widehat{X}}^{(\widehat{X}_{\widehat{T}})} \subset \Lambda_{G,X}^{(X_T)} \cap \widehat{\mathfrak{t}}.$$

Proof. $\widehat{X}_{\widehat{T}}$ was constructed the proof of [Lo7], Proposition 4.6. Recall the construction briefly.

In the notation of Proposition 4.3.4 set $\widehat{X}' := \widehat{X} \times V^H$. By Proposition 4.2.6, there is a connected saturated neighborhood O of x in X_M that is equivariantly symplectomorphic to an open saturated neighborhood of the base point x' in $\widehat{X}'$. Denote the last neighborhood also by O. Moreover, we may assume that the intersection of O with any $\widehat{T}$-cross-section of $\widehat{X}'$ is connected. Let $\widehat{X}'_{\widehat{T}}$ be a $\widehat{T}$-cross-section of $\widehat{X}'$ containing a connected component of $O \cap X_T$. By Proposition 4.6 from [Lo7], we get (4.2). The remaining part of assertion 1 was proved in [Lo7], Proposition 5.3.

By [Lo6], Lemma 6.10, $\mathfrak{X}_{M,X_M}^{(X_T)} = \mathfrak{X}_{G,X}^{(X_T)}, \Lambda_{M,X_M}^{(X_T)} \subset \Lambda_{G,X}^{(X_T)}$. Tautologically, the intersection of the inefficiency kernel for the action $T : X_T$ with $\widehat{G}$ coincides with the inefficiency kernel for the action $T \cap \widehat{G} : X_T$. So it follows directly from the definition of $\mathfrak{X}_{\bullet, \bullet}^{(\bullet)}$ that $\mathfrak{X}_{\widehat{G}, X_M}^{(X_T)} = p(\mathfrak{X}_{M,X_M}^{(X_T)})$. Further, by [Lo6], Lemma 6.14, $\Lambda_{\widehat{G}, X_M}^{(X_T)} = \Lambda_{M,X_M}^{(X_T)}$. Since O admits open embeddings into both $\widehat{X}', X$, we get $\mathfrak{X}_{\widehat{G}, \widehat{X}'}^{(\widehat{X}'_{\widehat{T}})} = \mathfrak{X}_{\widehat{G}, X_M}^{(X_T)}$. This proves assertion 2. To prove the third assertion we may (and will) assume that $X_M = X, \widehat{G} = G$. In this case we need to check that $\Lambda_{G, \widehat{X}'}^{(\widehat{X}'_T)} \subset \Lambda_{G,X}^{(X_T)}$ or, equivalently, $\mathfrak{A}_{G, \widehat{X}'}^{(\widehat{X}'_T)} \supset \mathfrak{A}_{G,X}^{(X_T)}$.

Choose $\varphi \in \mathfrak{A}_{G,X}^{(\cdot)}$. Any component component of $O \cap X_T$ is T-stable whence φ-stable. Therefore O and $\widehat{X}'_T \cap O$ are φ-stable. It remains to check that there is an (automatically unique) element $\overline{\varphi} \in \mathfrak{A}_{G, \widehat{X}'}^{(\cdot)}$ such that $\overline{\varphi}|_O = \varphi|_O$. (4.2) and Lemma 5.6 from [Lo6] yield $\mathfrak{A}_{G,X^{pr}}^{(X_T)} \subset \mathfrak{A}_{G, \widehat{X}'^{pr}}^{(\widehat{X}'_T)}$, so one can find an element $\overline{\varphi} \in \mathfrak{A}_{\widehat{G}, \widehat{X}'^{pr}}^{(\cdot)}$ with $\overline{\varphi}|_{O \cap \widehat{X}'_T} = \varphi|_{O \cap \widehat{X}'_T}$. Therefore $\overline{\varphi}|_{O \cap \widehat{X}'^{pr}} = \varphi|_{O \cap \widehat{X}'^{pr}}$. So the rational mapping $\overline{\varphi} : \widehat{X}' \dashrightarrow \widehat{X}'$ is defined in all divisors intersecting O. But all components of $\widehat{X}' \setminus \widehat{X}'^{pr}$ are $\mathbb{C}^\times$-stable. Therefore any of them intersects O. It follows that $\overline{\varphi}$ is a morphism. Applying [Lo6], Lemma 5.6, we complete the proof. $\square$

4.4. **Some properties of root lattices of affine Hamiltonian varieties.** In this subsection X is an untwisted conical affine Hamiltonian G-variety such that $m_G(X) = \dim G$.

We assume, in addition, that X is locally orthogonalizable in the sense of the following definition.

Definition 4.4.1. A smooth affine G-variety Y is called *locally orthogonalizable* (shortly, l.o.), if for any $y \in Y$ with closed G-orbit the G_y-module $T_y Y$ is orthogonal.

Remark 4.4.2. The G_y-module $T_y Y$ is orthogonal iff the slice module $T_y Y / \mathfrak{g}_* y$ is. This stems easily from the fact that the G_y-module $\mathfrak{g}_* y \cong (\mathfrak{g}_y)^\perp$ is an orthogonal submodule of $\mathfrak{g}$.

The following simple lemma provides some examples of l.o. G-varieties.

Lemma 4.4.3.   (1) *Let H be a reductive subgroup of G and V be an H-module. Then the G-variety $G *_H V$ is l.o. iff the H-module V is orthogonal.*
  (2) *Let Y be a l.o. G-variety and $y \in Y$ a point with closed G-orbit. Then $G *_{G_y}(T_y Y / \mathfrak{g}_* y)$ is l.o.*
  (3) *Let X_0 be a smooth affine G-variety. Then $T^* X_0$ is a l.o. G-variety.*
  (4) *If X is a l.o. G-variety and G^0 is a subgroup of G containing (G, G), then X is l.o. as a G^0-variety.*

Proof. In the notation of assertion 1, note that the $H_y = G_y$-modules $T_y(G *_H V), \mathfrak{g}/\mathfrak{h} \oplus V$ are isomorphic, where $y = [1, v] \in G *_H V$. Assertion 1 follows from Remark 4.4.2. The second assertion follows from the Luna slice theorem. In assertion 3 note that for any $y \in T^* X_0$ with closed G-orbit the G_y-module $T_y(T^* X_0)$ is isomorphic to $T_{\pi(y)} X_0 \oplus T^*_{\pi(y)} X_0$, where $\pi : T^* X_0 \to X_0$ is the canonical projection. To prove assertion 4 note that $G^0 x$ is closed provided Gx is. $\square$

We also need the notion of a $\mathfrak{g}$-stratum introduced in [Lo7].

Definition 4.4.4. A pair $(\mathfrak{h}, V)$, where $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$ and V is an $\mathfrak{h}$-module, is said to be a $\mathfrak{g}$-stratum. Two $\mathfrak{g}$-strata $(\mathfrak{h}_1, V_1), (\mathfrak{h}_2, V_2)$ are called *equivalent* if there exists $g \in G$ and a linear isomorphism $\varphi : V_1/V_1^{\mathfrak{h}_1} \to V_2/V_2^{\mathfrak{h}_2}$ such that $\mathrm{Ad}(g)\mathfrak{h}_1 = \mathfrak{h}_2$ and $(Ad(g)\xi)\varphi(v_1) = \varphi(\xi v_1)$ for all $\xi \in \mathfrak{h}_1, v_1 \in V_1/V_1^{\mathfrak{h}_1}$.

Definition 4.4.5. Let Y be a smooth affine variety and $y \in Y$ a point with closed G-orbit. The pair $(\mathfrak{g}_y, T_y Y / \mathfrak{g}_* y)$ is called the $\mathfrak{g}$-stratum of y. We say that $(\mathfrak{h}, V)$ is a $\mathfrak{g}$-stratum of Y if $(\mathfrak{h}, V)$ is equivalent to a $\mathfrak{g}$-stratum of a point of Y. In this case we write $(\mathfrak{h}, V) \rightsquigarrow_\mathfrak{g} Y$.

For $\alpha \in \Delta(\mathfrak{g})$ define $\mathfrak{g}$-strata $S^{(\alpha)}, R^{(\alpha)}$ as follows: $S^{(\alpha)} = (\mathfrak{g}^{(\alpha)}, \mathbb{C}^2 \oplus \mathbb{C}^2) R^{(\alpha)} = (\mathbb{C}\alpha^\vee, V)$, where V is the two-dimensional $\mathbb{C}\alpha^\vee$-module, where $\alpha^\vee$ acts with weights ± 1.

All results concerning the root lattice of X are based on the following proposition.

Proposition 4.4.6. *Let us fix a T-section X_T of X. Let $\alpha \in \Delta(\mathfrak{g})$ be such that $\alpha \notin \Lambda_{G,X}^{(X_T)}$ but $s_\alpha \in W_{G,X}^{(X_T)}$. Then $R^{(\alpha)} \rightsquigarrow_\mathfrak{g} X, 2\alpha \in \Lambda_{G,X}^{(X_T)}$.*

Proof. Set $\mathfrak{m} = \mathfrak{t} + \mathfrak{g}^{(\alpha)}$. This is a Levi subalgebra in $\mathfrak{g}$. Denote by M the corresponding Levi subgroup of G. Applying Proposition 4.3.4 to M, we find a point $x \in X$ satisfying conditions (a)-(e) of this proposition. Set $\widehat{G} = G^{(\alpha)}$ and let $\widehat{X}$ be the coisotropic model variety defined in condition (d) of Proposition 4.3.4. By Proposition 4.3.5, $\alpha \notin \Lambda_{\widehat{G}, \widehat{X}}^{(\cdot)}$, $\widehat{X}$ is untwisted as a Hamiltonian $\widehat{G}$-variety and $s_\alpha \in W_{\widehat{G}, \widehat{X}}^{(\cdot)}$. Besides, by Lemma 4.4.3, the $\widehat{G}$-variety $\widehat{X}$ is l.o. Our

claim will follow if we check that $\widehat{X} = T^*(\widehat{G}/F)$, where F° is a maximal torus in $\widehat{G}$. Indeed, from Theorem 4.3.2 one can easily deduce that $2\alpha \in \Lambda^{(\cdot)}_{\widehat{G},\widehat{X}}$ and, thanks to Proposition 4.3.5, $2\alpha \in \Lambda^{(X_T)}_{G,X}$. Below we assume that $G = \widehat{G} \cong \mathrm{SL}_2, X = \widehat{X}$.

Let H, η, V be such that $X = M_G(H, \eta, V)$. Since X is coisotropic, we see that $\dim X = \dim G + \mathrm{rk}\, G = 4$. If $\eta \neq \{0\}$, then H is discrete and $V = \{0\}$. One easily verifies that in this case $\Lambda^{(\cdot)}_{G,X} = \Lambda(\mathfrak{g})$. Thus $\eta = 0$ and either H° is a maximal torus of G or $H = \widehat{G}$. It remains to consider the case $H = G$. Here $\dim V = 4$. Since V is both orthogonal and symplectic, we see that V is the direct sum of two copies of the tautological SL_2-module. But in this case $W^{(\cdot)}_{G,X} = \{1\}$. $\qquad\square$

Our next task is to classify coisotropic model varieties with some special properties.

Proposition 4.4.7. *Suppose $G \cong \mathrm{SL}_3$ and $X = M_G(H, \eta, V)$, where $\mathrm{cork}_G(X) = 0, \eta$ is nilpotent. Then the following conditions are equivalent:*

  (1) $\Lambda^{(\cdot)}_{G,X} \neq \Lambda(\mathfrak{g})$.
  (2) $\eta = 0$ and the pair $(\mathfrak{h}, V)$ is indicated in Table 4.1.

Under these equivalent conditions, $\Lambda^{(\cdot)}_{G,X}$ depends (up to $W(\mathfrak{g})$-conjugacy) only on the pair $(\mathfrak{h}, V)$. Generators of $\Lambda^{(\cdot)}_{G,X}$ are presented in the third column of Table 4.1.

In the second row of Table 4.1 $\mathbb{C}_{\pm\chi}$ denotes the one-dimensional $\mathfrak{h}$-module corresponding to a nonzero character χ of $\mathfrak{h}$.

Table 4.1: Coisotropic model varieties with small root lattices for $G = \mathrm{SL}_3$

N	$(\mathfrak{h}, V)$	$\Lambda^{(\cdot)}_{G,X}$
1	$(\mathfrak{sl}_2, 0)$	$\varepsilon_1 - \varepsilon_3$
2	$(\mathfrak{sl}_2 \times \mathbb{C}, \mathbb{C}_\chi \oplus \mathbb{C}_{-\chi})$	$\varepsilon_1 - \varepsilon_3$
3	$(\mathfrak{so}_3, 0)$	$2\alpha_1, 2\alpha_2$

Proof. Suppose $\Lambda^{(\cdot)}_{G,X} \neq \Lambda(\mathfrak{g})$. Since $S^{(\alpha)} \rightsquigarrow_\mathfrak{g} X$ or $R^{(\alpha)} \rightsquigarrow_\mathfrak{g} X$, we see that there is $x \in \mathfrak{h}, x \sim_G \alpha^\vee$, where $\alpha \in \Delta(\mathfrak{g})$. It follows that $\mathfrak{g}^H$ does not contain a nilpotent element whence $\eta = 0$ and V is an orthogonal H-module. Therefore there is a H-module V_0 such that $V \cong V_0 \oplus V_0^*$. As we have seen in [Lo1], it follows that the Hamiltonian G-varieties X and T^*X_0, where $X_0 = G *_H V$, are isomorphic. In particular, $\Lambda^{(\cdot)}_{G,X} \sim_{W(\mathfrak{g})} \Lambda_{G,X_0}$, so the former lattice depends only on $(\mathfrak{h}, V)$. Since $\mathrm{cork}_G(X) = 0$, we get $2(\dim \mathfrak{g} - \dim \mathfrak{h}) + 2\dim V_0 = \mathrm{rk}\, \mathfrak{g} + \dim \mathfrak{g}$, equivalently,

$$(4.5) \qquad\qquad\qquad\qquad \dim \mathfrak{h} - \dim V_0 = 3.$$

At first, consider the case $S^{(\alpha)} \rightsquigarrow_\mathfrak{g} X$. Proposition 5.2.1 from [Lo5] implies $[\mathfrak{h}, \mathfrak{h}] \sim_G \mathfrak{g}^{(\alpha)}, V = V^{[\mathfrak{h},\mathfrak{h}]}$. From (4.5) it follows that $(\mathfrak{h}, V)$ is one of pairs 1,2 of Table 4.1. Let us check that $\Lambda^{(\cdot)}_{G,X}$ coincides with the lattice indicated in Table 4.1. By Proposition 3.2.7, if V_0 is a nontrivial 1-dimensional $\mathfrak{h}$-module, then $\Lambda_{G,G*_H V_0} = \Lambda(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}])$. To determine the last lattice we use [Krä], Tabelle 1, and Proposition 5.2.1.

Now consider the case $S^{(\alpha)} \not\rightsquigarrow_\mathfrak{g} X$. In this case $W^{(\cdot)}_{G,X} = W(\mathfrak{g})$ and $R^{(\alpha)} \rightsquigarrow_\mathfrak{g} X$. Since $\mathfrak{g}$ has no spherical modules of rank 2, see [Le], we get $\mathfrak{h} \neq \mathfrak{g}$. It follows that $\dim \mathfrak{h} \leqslant 4$ and, in the

case of equality, $\dim V_0 = 1$, so we get pair N2. So $\dim \mathfrak{h} = 3, V_0 = \{0\}$ whence $\mathfrak{h} = \mathfrak{so}_3$. By [Krä], Tabelle 1, $\mathfrak{X}_{\mathrm{PSL}_3,\mathrm{PO}_3} = 2\Lambda(\mathfrak{g})$. Applying Proposition 5.2.1, we get $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$. $\square$

Corollary 4.4.8. *Let $\mathfrak{g}$ be a simple Lie algebra with $\mathrm{rk}\,\mathfrak{g} > 2$. Denote by X_T a T-cross-section of X. Let $\alpha, \alpha_1 \in \Delta(\mathfrak{g})$ be such that $s_\alpha \in W_{G,X}^{(X_T)}, s_{\alpha_1} \notin W_{G,X}^{(X_T)}$ and $\mathfrak{g}^{(\alpha,\alpha_1)} \cong \mathfrak{sl}_3$. Then $\alpha \in \Lambda_{G,X}^{(X_T)}$.*

Proof. Set $\mathfrak{m} := \mathfrak{t} + \mathfrak{g}^{(\alpha,\alpha_1)}$. Since $\mathfrak{g} \not\cong G_2$, we see that $\mathfrak{m}$ is a Levi subalgebra in $\mathfrak{g}$. Let M denote the corresponding Levi subgroup. Apply Proposition 4.3.4 to M. Let x be a point in X satisfying the conditions (a)-(e) of this proposition, $\widehat{G} := (M, M), \widehat{T} := T \cap (M, M)$, and $\widehat{X}$ be the model variety defined in condition (d). Thanks to Proposition 4.3.5, there is a $\widehat{T}$-cross-section $\widehat{X}_{\widehat{T}}$ of $\widehat{X}$ such that

$$(4.6) \qquad W_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})} = W_{G,X}^{(X_T)} \cap M/T,$$

$$(4.7) \qquad \Lambda_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})} \subset \Lambda_{G,X}^{(X_T)}.$$

(4.6) and [Lo7], Corollary 4.16, imply that $W_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})}$ is generated by s_α. By (4.7), $\Lambda_{\widehat{G},\widehat{X}}^{(\cdot)} \neq \Lambda(\widehat{\mathfrak{g}})$. From Proposition 4.4.7 it follows that $\Lambda_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})}$ is spanned by α. To complete the proof apply (4.7) one more time. $\square$

Let us introduce one more $\mathfrak{g}$-stratum. Let $\alpha_1, \alpha_2 \in \Delta(\mathfrak{g})$ be such that $\mathfrak{g}^{(\alpha_1,\alpha_2)} \cong \mathfrak{sl}_3$. By $\widetilde{R}^{(\alpha_1,\alpha_2)}$ we denote the $\mathfrak{g}$-stratum $(\mathfrak{s}, V)$, where $\mathfrak{s} = \mathfrak{so}_3 \subset \mathfrak{sl}_3 \cong \mathfrak{g}^{(\alpha_1,\alpha_2)}$ and V is the 5-dimensional irreducible $\mathfrak{s}$-module.

Proposition 4.4.9. *Let $\mathfrak{g}, X_T$ be such as in Corollary 4.4.8 and α_1, α_2 such as in the definition of $\widetilde{R}^{(\alpha_1,\alpha_2)}$. Suppose $s_\alpha \in W_{G,X}^{(X_T)}$ for all $\alpha \in W(\mathfrak{g})\alpha_1$. Then the following conditions are equivalent:*

(1) $\widetilde{R}^{(\alpha_1,\alpha_2)} \rightsquigarrow_{\mathfrak{g}} X$.
(2) $\alpha_1 \notin \Lambda_{G,X}^{(X_T)}, 2\alpha_1 \in \Lambda_{G,X}^{(X_T)}$.

Proof. (2) $\Rightarrow$ (1). Analogously to the proof of Corollary 4.4.8, we reduce the proof to the situation $G = \mathrm{SL}_3, X = M_G(H, \eta, V), \mathrm{cork}_G(X) = 0, \eta$ is nilpotent. We have $W_{G,X}^{(\cdot)} = W(\mathfrak{g})$. Since $\Lambda_{G,X}^{(X_T)}$ is $W_{G,X}^{(X_T)}$-stable (Lemma 4.3.3), we see that all three inclusions $\alpha_1 \in \Lambda_{G,X}^{(X_T)}, \alpha_2 \in \Lambda_{G,X}^{(X_T)}, \alpha_1 + \alpha_2 \in \Lambda_{G,X}^{(X_T)}$ are equivalent. (1) follows now from Proposition 4.4.7.

(1) $\Rightarrow$ (2). Assume that (2) does not hold. Note that $s_\alpha \in W_{G,X}^{(X_T)}, \alpha \in \Lambda_{G,X}^{(X_T)}$ for all $\alpha \in \Delta(\mathfrak{g})$ of the same length with α_1. Indeed, by the choice $\mathfrak{g}$ and α_1, there are $\alpha^1, \ldots, \alpha^k \in W(\mathfrak{g})\alpha_1$ such that $s_{\alpha^k} \ldots s_{\alpha^1}\alpha = \alpha_1$. But $s_{\alpha^j} \in W_{G,X}^{(X_T)}, j = \overline{1, k}$ and $\Lambda_{G,X}^{(X_T)}$ is $W_{G,X}^{(X_T)}$-stable.

Let $Y \subset X$ be the set of all points $x \in X$ such that Gx is closed and the $\mathfrak{g}$-stratum of x is equivalent to $\widetilde{R}^{(\alpha_1,\alpha_2)}$. Thanks to the Luna slice theorem, Y is a locally closed subvariety of X and $\overline{Y}//G = \overline{Y//G}$ is a subvariety of pure codimension 2 in $X//G$.

There is a point $x \in Y$ such that $\mathfrak{g}_x \subset \mathfrak{g}^{(\alpha_1,\alpha_2)}$. Let us check that the subspace $\ker \alpha_1 \cap \ker \alpha_2 \subset \mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_x)$. Indeed, the inclusion $\ker \alpha_1 \cap \ker \alpha_2 \subset \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_x)$ stems from $\mathfrak{g}_x \subset \mathfrak{g}^{(\alpha_1,\alpha_2)}$. On the other hand, $\mathrm{rk}\,\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_x) < \mathrm{rk}\,\mathfrak{g} - 1$, because $\mathfrak{g}_x \not\sim_G \mathfrak{g}^{(\beta)}$ for any $\beta \in \Delta(\mathfrak{g})$.

By the previous paragraph, $\psi_{G,X}(Y) \subset \pi_{W(\mathfrak{g}),\mathfrak{t}}(\ker \alpha_1 \cap \ker \alpha_2)$. By [Lo2], Theorem 1.2.3, $(\psi_{G,X}//G)^{-1}(\pi_{W,\mathfrak{t}}(\ker \alpha_1 \cap \ker \alpha_2))$ has pure codimension 2 in $X//G$ and $\psi_{G,X}(Y)$ is dense in $\pi_{W(\mathfrak{g}),\mathfrak{t}}(\ker \alpha_1 \cap \ker \alpha_2)$.

Replacing x with gx for appropriate $g \in G$, we get $\mu_{G,X}(x)_s \in \mathfrak{z}(\mathfrak{m}) \cap \mathfrak{m}^{pr}$ (here automatically $\mathfrak{g}_x \subset \mathfrak{g}^{(\alpha_1,\alpha_2)}$). Then $\widehat{\psi}_{G,X}(x) \in \pi_{W_{G,X}^{(\cdot)},\mathfrak{a}_{G,X}^{(\cdot)}}(\mathfrak{z}(\mathfrak{m}_0) \cap \mathfrak{m}_0^{pr})$ for some Levi subalgebra $\mathfrak{m}_0 \subset \mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{m}_0, \mathfrak{m}_0 \sim_G \mathfrak{m}$. So, replacing $\mathfrak{m}$ with $\mathfrak{m}_0$ if necessary, we may assume that $\widehat{\psi}_{G,X}(x) \in \pi_{W_{G,X}^{(\cdot)},\mathfrak{a}_{G,X}^{(\cdot)}}(\mathfrak{z}(\mathfrak{m}) \cap \mathfrak{m}^{pr})$.

Now let X'_T be a T-cross-section of X. There exists $w \in W$ such that $wW_{G,X}^{(X_T)}w^{-1} = W_{G,X}^{(X'_T)}, w\Lambda_{G,X}^{(X_T)} = \Lambda_{G,X}^{(X'_T)}$. By the first paragraph of the proof of $(1) \Rightarrow (2)$, $s_{\alpha_1}, s_{\alpha_2} \in W_{G,X}^{(X'_T)}$, $\alpha_1 \in \Lambda_{G,X}^{(X'_T)}$. Replacing X_T with X'_T, if necessary, we may assume that X_T is contained in a unique M-section of X containing x. So x satisfies conditions (a)-(e) of Proposition 4.3.4. Set $\widehat{G} := (M, M)$ and let $\widehat{X}$ be the model variety constructed in (d). By the choice of x, we get $\widehat{X} = T^*(\mathrm{SL}_3/H)$ with $\mathfrak{h} = \mathfrak{so}_3$. So $W_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})} = W(\widehat{\mathfrak{g}}), \Lambda_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})} = 2\Lambda(\widehat{\mathfrak{g}})$ (Proposition 4.4.7).

Applying assertion 2 of Proposition 4.3.5, we see that for any $w \in W(\mathfrak{g})$ the projection of $w\alpha_1$ to $\widehat{\mathfrak{t}} := \mathfrak{t} \cap [\mathfrak{m}, \mathfrak{m}]$ lies in $\mathfrak{X}_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})}$ for an appropriate $\widehat{T}$-cross-section $\widehat{X}_{\widehat{T}}$ of $\widehat{X}$. But such projections span the lattice $\mathfrak{X}_{\mathrm{SL}_3}$. By Lemma 4.3.3, $w\xi - \xi \in \Lambda_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})}$ for any $\xi \in \Lambda$, $w \in W_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})}$. Note that $s_{\alpha_1}(\pi_1) - \pi_1 = -\alpha_1$. Contradiction with $\Lambda_{\widehat{G},\widehat{X}}^{(\widehat{X}_{\widehat{T}})} = 2\Lambda(\widehat{\mathfrak{g}})$. $\square$

5. Computation of root and weight lattices for affine homogeneous spaces

5.1. **Introduction.** In this section G is a connected reductive group, $B \subset G$ is its Borel subgroup and $T \subset B$ is its maximal torus. Throughout the section $X = G/H$ is an affine homogeneous space of rank $\mathrm{rk}(G)$. Our objective is to compute the lattices $\Lambda_{G,X}, \mathfrak{X}_{G,X}$. At first, we compute the root lattices and then, using this computation, determine the weight lattices.

Recall that the root lattice $\Lambda_{G,G/H}$ depends only on the pair $(\mathfrak{g}, \mathfrak{h})$ (by Proposition 3.2.6) so we write $\Lambda(\mathfrak{g}, \mathfrak{h})$ instead of $\Lambda_{G,G/H}$. Lemma 3.2.8 implies that $\Lambda(\mathfrak{g}, \mathfrak{h}) \subset \Lambda(\mathfrak{g}, \mathfrak{h}_1)$ for any ideal $\mathfrak{h}_1 \subset \mathfrak{h}$.

Now let $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \bigoplus_{i=1}^k \mathfrak{g}_i$ be the decomposition into the direct sum of the center and simple ideals, and $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$. By Proposition 3.2.12, $\Lambda(\mathfrak{g}, \mathfrak{h}) = \bigoplus_{i=1}^k \Lambda(\mathfrak{g}_i, \mathfrak{h}_i)$. So it is enough to compute $\Lambda(\mathfrak{g}, \mathfrak{h})$ for simple $\mathfrak{g}$.

Definition 5.1.1. Let $\mathfrak{g}$ be simple. A reductive subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called Λ-essential if $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{t}$ and for any ideal $\mathfrak{h}_1 \subset \mathfrak{h}$ the inclusion $\Lambda(\mathfrak{g}, \mathfrak{h}) \subset \Lambda(\mathfrak{g}, \mathfrak{h}_1)$ is strict.

The following theorem is the main result on the computation of $\Lambda(\mathfrak{g}, \mathfrak{h})$.

Theorem 5.1.2.   (1) *All Λ-essential subalgebras $\mathfrak{h} \subset \mathfrak{g}$ together with the corresponding lattices $\Lambda(\mathfrak{g}, \mathfrak{h})$ are presented in 5.1.*
   (2) *For any reductive subalgebra $\mathfrak{h} \subset \mathfrak{g}$ there is a unique ideal $\mathfrak{h}^{\Lambda-ess} \subset \mathfrak{h}$ such that $\mathfrak{h}^{\Lambda-ess}$ is Λ-essential and $\Lambda(\mathfrak{g}, \mathfrak{h}) = \Lambda(\mathfrak{g}, \mathfrak{h}^{\Lambda-ess})$. Such $\mathfrak{h}^{\Lambda-ess}$ is maximal (w.r.t inclusion) among all ideals in $\mathfrak{h}$ that are Λ-essential subalgebras in $\mathfrak{g}$.*

Table 5.1: Λ-essential subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and lattices
$\Lambda(\mathfrak{g}, \mathfrak{h})$

N	$\mathfrak{g}$	$\mathfrak{h}$	$\Lambda(\mathfrak{g}, \mathfrak{h})$
1	$\mathfrak{sl}_n, n \geqslant 2$	$\mathfrak{so}_n$	$2\Lambda(\mathfrak{g})$
2	$\mathfrak{sl}_{2n+1}$	$\mathfrak{sl}_{n+1}$	$\{\sum_{i \neq n+1} x_i \varepsilon_i \mid x_i \in \mathbb{Z}, \sum_{i \neq n+1} x_i = 0\}$
3	$\mathfrak{sl}_{2n+1}$	$\mathfrak{sp}_{2n}$	$\{\sum x_i \varepsilon_i \mid x_i \in \mathbb{Z}, \sum_i x_{2i} = \sum_i x_{2i+1} = 0\}$
4	$\mathfrak{so}_{2n+1}, n \geqslant 3$	$\mathfrak{so}_{n+1}$	$\{\sum_{i=1}^n x_i \varepsilon_i \mid x_i \in \mathbb{Z}, \sum_{i=1}^n x_i \equiv 0 (\mathrm{mod}\, 2)\}$
5	$\mathfrak{so}_{2n+1}, n \geqslant 3$	$\mathfrak{so}_{n+1} \oplus \mathfrak{so}_n$	$2\Lambda(\mathfrak{g})$
6	$\mathfrak{so}_{2n+1}, n \geqslant 3$	$\mathfrak{gl}_n$	$\{\sum_{i=1}^n x_i \varepsilon_i \mid x_i \in \mathbb{Z}, \sum_{i=1}^{[n/2]} x_{n-2i} \equiv 0 (\mathrm{mod}\, 2)\}$
7	$\mathfrak{sp}_{2n}, n \geqslant 2$	$\mathfrak{sl}_n$	$\{2\sum_{i=1}^n x_i \varepsilon_i \mid x_i \in \mathbb{Z}\}$
8	$\mathfrak{sp}_{2n}, n \geqslant 2$	$\mathfrak{gl}_n$	$2\Lambda(\mathfrak{g})$
9	$\mathfrak{so}_{2n}, n \geqslant 4$	$\mathfrak{so}_n \oplus \mathfrak{so}_n$	$2\Lambda(\mathfrak{g})$
10	G_2	$A_1 \times \widetilde{A}_1$	$2\Lambda(\mathfrak{g})$
11	F_4	C_3	$\{\sum_{i=1}^4 x_i \varepsilon_i \mid x_i \in \mathbb{Z}, \sum_{i=1}^4 x_i \equiv 0 (\mathrm{mod}\, 2)\}$
12	F_4	$C_3 \times A_1$	$2\Lambda(\mathfrak{g})$
13	E_6	C_4	$2\Lambda(\mathfrak{g})$
14	E_7	A_7	$2\Lambda(\mathfrak{g})$
15	E_8	D_8	$2\Lambda(\mathfrak{g})$

Proceed to computing weight lattices. Till the end of the subsection G is an arbitrary connected reductive group. By $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ we denote all simple ideals of $\mathfrak{g}$.

Let us introduce some notation. Let H be a reductive subgroup of G with $\mathrm{rk}_G(G/H) = \mathrm{rk}\, G$. Let $\widehat{H}$ denote the connected subgroup in G with Lie algebra $\bigoplus_{i=1}^k (\mathfrak{h} \cap \mathfrak{g}_i)^{\Lambda-ess}$. By $H^{\mathfrak{X}-sat}$ we denote the inverse image of $\mathfrak{A}_{G,G/\widehat{H}} \subset N_G(\widehat{H})/\widehat{H}$ in $N_G(\widehat{H})$. It follows directly from the definition that $\mathfrak{X}_{G,G/H^{\mathfrak{X}-sat}} = \Lambda(\mathfrak{g}, \mathfrak{h})$. Here is the main result concerning the computation of weight lattices.

Theorem 5.1.3. *Let H be as in the previous paragraph. Suppose G is algebraically simply connected (i.e., is the direct product of a torus and a simply connected semisimple group). Set $H_0 := H \cap H^{\mathfrak{X}-sat}$.*

(1) *$\mathfrak{X}_{G,G/H} = \mathfrak{X}_{G,G/H_0}$. Further, $\mathfrak{h}_0 = \mathfrak{h}^{\Lambda-ess}$ and H_0 is the maximal normal subgroup of H contained in $H^{\mathfrak{X}-sat}$. Finally, $H_0^{\mathfrak{X}-sat} = H^{\mathfrak{X}-sat}$.*

(2) *Suppose $H \subset H^{\mathfrak{X}-sat}$. Recall that there is the natural duality $\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g}, \mathfrak{h})$ $H^{\mathfrak{X}-sat}/H^\circ = \mathfrak{A}_{G,G/H^\circ}$. When G is simple this duality is described in Remark 5.1.4 below. In the general case there are the equalities*

$$\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g}, \mathfrak{h}) = \mathfrak{X}_{Z(G)^\circ} \oplus \bigoplus_{i=1}^k \mathfrak{X}_{G_i, G_i/H_i^\circ}/\Lambda(\mathfrak{g}_i, \mathfrak{h}_i),$$

(5.1)

$$H^{\mathfrak{X}-sat}/H^\circ = Z(G)^\circ \times \prod_{i=1}^k H_i^{\mathfrak{X}-sat}/H_i^\circ.$$

The duality between $\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g}, \mathfrak{h})$ $H^{\mathfrak{X}-sat}/H^\circ$ is the direct product of the dualities between the corresponding factors in (5.1).

(3) *If $H \subset H^{\mathfrak{X}-sat}$, then $\mathfrak{X}_{G,G/H}$ coincides with the inverse image of the annihilator of H/H° in $\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g}, \mathfrak{h})$ under the natural epimorphism $\mathfrak{X}_{G,G/H^\circ} \twoheadrightarrow \mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g}, \mathfrak{h})$.*

Remark 5.1.4. This remark gives a more or less explicit description of the duality between $\mathfrak{X}_{G,G/H^\circ}/\mathfrak{X}_{G,G/H^{\mathfrak{X}-sat}}$ and $H^{\mathfrak{X}-sat}/H^\circ$, where G is a simple group and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ indicated in Table 5.1. By χ_λ we denote the character of $H^{\mathfrak{X}-sat}/H^\circ$ corresponding to a weight $\lambda \in \mathfrak{X}_{G,G/H^\circ}$. Note, at first, that the center of G is identified with $(\mathfrak{X}_G/\Lambda(\mathfrak{g}))^*$. Further, there is the natural homomorphism $\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g},\mathfrak{h}) \to \mathfrak{X}_G/\Lambda(\mathfrak{g})$. So any element of $Z(G)$ determines an element in $(\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g},\mathfrak{h}))^*$. Note that $Z(G) \subset H^{\mathfrak{X}-sat}$. Below we will see that the image of the map $Z(G) \to (\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g},\mathfrak{h}))^*$ equals $Z(G) \cap H^\circ$, and the corresponding map $\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g},\mathfrak{h}) \to \mathfrak{X}(Z(G)/Z(G) \cap H^\circ)$ coincides with $\lambda \mapsto \chi_\lambda$. Therefore it is enough to determine:

(1) the lattice $\mathfrak{X}_{G,G/H^\circ}$,
(2) elements from $H^{\mathfrak{X}-sat}$ whose images in $H^{\mathfrak{X}-sat}/Z(G)H^\circ$ generate the last group,
(3) characters χ_λ for these elements λ.

There are no elements as in (2) precisely for the pairs $(\mathfrak{g},\mathfrak{h})$ NN10-13,15. In all remaining cases the indicated information is presented in Table 5.2. In the first column the number of the pair in Table 5.1 is given. In the second column we indicate an element $\lambda \in \mathfrak{X}_{G,G/H^\circ}$ whose image in $\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g},\mathfrak{h})$ generates this group. In the square brackets the order of the image is given. Column 3 contains the group $H^{\mathfrak{X}-sat}/H^\circ$. If this group is finite, then we indicate a generator of $H^{\mathfrak{X}-sat}$. If the group is infinite (the pairs NN2,3), then we give a typical element of a subgroup complementing H° in $H^{\mathfrak{X}-sat}$. Finally, in the last column we indicate the character χ_λ for the element λ presented in column 2.

When $\operatorname{rk} H = \operatorname{rk} G$ (whence $Z(G) \subset H$ for any G) we consider the most convenient for us group G. In row 4 we assume that $G = SO_{2n+1}$, since $Z(G) \subset H^\circ$ for simply connected G too. In all remaining cases we assume that G is simply connected.

Table 5.2: Correspondence between $\mathfrak{X}(H^{\mathfrak{X}-sat}/H^\circ)$ and $\mathfrak{X}_{G,G/H^\circ}/\Lambda(\mathfrak{g},\mathfrak{h})$

N	λ	h	$\chi_\lambda(h)$
1	$2\pi_1[n]$	$diag(e^{i\pi/n},\ldots,e^{i\pi/n})d, d \in \varnothing(n) \setminus SO(n)$	$e^{-2\pi i/n}$
2	$\pi_1[\infty]$	$diag(t^{n+1},\ldots,t^{n+1},t^{-n},\ldots,t^{-n})$	t^{-n-1}
3	$\pi_2[\infty]$	$diag(t^{2n},t^{-1},\ldots,t^{-1})$	t^2
4	$\pi_1[2]$	$diag(-1,\ldots,-1,d), d \in \varnothing(n+1), \det(d) = (-1)^n$	-1
5	$2\pi_n[2]$	$h \in N_G(\mathfrak{h}) \setminus H^\circ$	-1
6	$\pi_2[2]$	$h \in N_G(\mathfrak{h}) \setminus H^\circ$	-1
7	$\pi_n[2]$	$\exp(\pi i \pi_n/2^m), 2^m|n, 2^{m+1} \nmid n$	-1
8	$2\pi_1[2]$	$h \in N_G(\mathfrak{h}) \setminus H^\circ$	-1
9	$2\pi_n[2]$	$h \in N_G(\mathfrak{h}) \setminus H^\circ$	-1
14	$2\pi_1$	$h \in N_G(\mathfrak{h}) \setminus H^\circ$	-1

Let us describe the structure of this section. In Subsection 5.2 we establish the equality of the root lattice $\Lambda(\mathfrak{g},\mathfrak{h})$ and the weight lattice $\mathfrak{X}_{G,G/\widetilde{H}}$ for a certain subgroup $\widetilde{H} \subset G$ constructed from H. In Subsection 4.4 we get some results on the structure of the root lattices for a certain class of affine Hamiltonian varieties. Subsections 5.3,5.4 are devoted to the proofs of Theorems 5.1.2,5.1.3. The former is based mostly on results of Subsection 4.4, the latter is quite easy. Finally in Subsection 5.5 we show how to find a point from the distinguished component of $(G/H)^{L_0 G,G/H}$.

5.2. **Connection between root and weight lattices of homogeneous spaces.** In this subsection G is a connected reductive group and H is its algebraic subgroup. Our goal in this subsection is to prove that $\Lambda_{G,G/H}$ coincides with $\mathfrak{X}_{G,G/\widetilde{H}}$, where $\widetilde{H}$ is a subgroup of $N_G(H)$ constructed from H.

The basic idea of the construction of $\widetilde{H}$ is that $\widetilde{H}/H \subset N_G(H)/H \cong \mathrm{Aut}^G(G/H)$ should contain all central automorphisms. Namely let Z denote the semisimple part of the center of $N_G(H)/H$. For $\widetilde{H}$ we take the inverse image of Z under the canonical epimorphism $N_G(H) \twoheadrightarrow N_G(H)/H$.

Proposition 5.2.1. $\Lambda_{G,G/H} = \mathfrak{X}_{G,G/\widetilde{H}}$.

Lemma 5.2.2. *Let X_0 be a G-variety and $T_0 \subset \mathrm{Aut}^G(X_0)$ a quasitorus. Further, let X_1 be a rational quotient for the action $T_0 : X_0$ equipped with an action of G such that the rational quotient mapping $X \to X_0$ is G-equivariant. Then $\mathfrak{X}_{G,X_1} \subset \mathfrak{X}(T)$ coincides with the annihilator of $\iota_{G,X_0}(T_0 \cap \mathfrak{A}_{G,X_0}) \subset A_{G,X_0}$ in $\mathfrak{X}_{G,X_0} \cong \mathfrak{X}(A_{G,X_0})$.*

Proof. Let us reduce the proof to the case when G is a torus. A standard argument, compare with the proof of Theorem 1.3 in [K2], shows that there are open B-stable quasiaffine subvarieties $X_0' \subset X_0, X_1' \subset X_1$. Embed X_0', X_1' to affine B-varieties $\overline{X}_0', \overline{X}_1'$ (this is possible by [PV], Theorem 1.4) and set $Z_i := \mathrm{Spec}(\mathbb{C}[\overline{X}_i']^U), i = 0,1$. Then $Z_i, i = 1,2$, is a rational quotient for the action $U : X_i$. Clearly, $\mathfrak{X}_{G,X_0} = \mathfrak{X}_{T,Z_0}, \mathfrak{X}_{G,X_1} = \mathfrak{X}_{T,Z_1}$. Since $\mathbb{C}(Z_1) \cong \mathbb{C}(X_0)^{U \times T_0}$, we see that Z_1 is a rational quotient for the action $T_0 : Z_0$. The action $T_0 : \mathbb{C}(X_0)^U$ is effective, for the action $T_0 : \mathbb{C}(X)$ is. It follows that the action $T_0 : Z_0$ is effective. For any $\lambda \in \mathfrak{X}_{G,X_0} = \mathfrak{X}_{T,Z_0}$ there is a T_0-isomorphism $\mathbb{C}(X_0)_\lambda^{(B)} \cong \mathbb{C}(Z_0)_\lambda^{(T)}$. Thus $T_0 \cap \mathfrak{A}_{G,X_0} = T_0 \cap \mathfrak{A}_{T,Z_0}$ and $\iota_{G,X_0}|_{T_0 \cap \mathfrak{A}_{G,X_0}} = \iota_{T,Z_0}|_{T_0 \cap \mathfrak{A}_{T,Z_0}}$. So it is enough to prove the claim of the lemma for the pair (T, Z_0) instead of (G, X_0). Further, we easily reduce to the case when the action $T : Z_0$ is effective.

Let us note that $L_{T,Z_1}, L_{T \times T_0, Z_0}$ coincide with the inefficiency kernels of the corresponding actions whence

(5.2) $$L_{T,Z_1} = \{t \in T | tz \in T_0 z \text{ for } z \in Z_0 \text{ in general position}\},$$

and

(5.3) $$L_{T \times T_0, Z_0} = (T \times T_0)_z.$$

for $z \in Z_0$ in general position. From (5.2) and (5.3) it follows that $L_{T,Z_1} = \pi_1(L_{T \times T_0, Z_0})$, where $\pi_1 : T \times T_0 \to T$ is the projection to the first factor. On the other hand, the action $T_0 : Z_0$ is effective whence the restriction of the projection $\pi_2 : T \times T_0 \to T_0$ to $L_{T \times T_0, Z_0}$ is an embedding. The image of this embedding coincides with $\mathfrak{A}_{T,Z_0} \cap T_0$, for it consists precisely of those elements of T_0 that act on Z_0 as elements of T. The homomorphism $\pi_1 \circ \pi_2^{-1}|_{T_0 \cap \mathfrak{A}_{T,Z_0}}$ maps an element $t_0 \in T_0 \cap \mathfrak{A}_{T,Z_0}$ to the element $t \in T$ such that $t_0 z = tz$ for all $z \in Z_0$. In other words, $\pi_1 \circ \pi_2^{-1}|_{T_0 \cap \mathfrak{A}_{T,Z_0}} = \iota_{T,Z_0}|_{T_0 \cap \mathfrak{A}_{T,Z_0}}$. Equivalently, $L_{T,Z_1} = \iota_{T,Z_0}(T_0 \cap \mathfrak{A}_{T,Z_0})$. $\square$

Proof of Proposition 5.2.1. Apply Lemma 5.2.2 to $X_0 := G/H, T_0 := Z_0, X_1 = G/\widehat{H}$. $\square$

5.3. **Proof of Theorem 5.1.2.** In this subsection $\mathfrak{g}$ is supposed to be simple. Let $\Delta(\mathfrak{g})^{min}$ denote the subsets of $\Delta(\mathfrak{g})$ consisting of all roots of minimal length and set $\Delta(\mathfrak{g})^{max} := \Delta(\mathfrak{g}) \setminus \Delta(\mathfrak{g})^{max}$.

Lemma 5.3.1. *Let $\mathfrak{h}$ be a nonzero Λ-essential subalgebra of $\mathfrak{g}$. Then*

(1) $\Delta(\mathfrak{g})^{min} \not\subset \Lambda(\mathfrak{g}, \mathfrak{h})$.
(2) If $\alpha \in \Delta(\mathfrak{g}) \setminus \Lambda(\mathfrak{g}, \mathfrak{h})$, then there is $h \in \mathfrak{h}$ such that $h \sim_G \alpha^\vee$ and

(5.4) $$\operatorname{tr}_{\mathfrak{g}} h^2 = 2 \operatorname{tr}_{\mathfrak{h}} h^2 + 8.$$

Proof. Assertion 1 stems from $\operatorname{Span}_{\mathbb{Z}}(\Delta(\mathfrak{g})^{min}) = \Lambda(\mathfrak{g})$. Proceed to assertion 2. By Proposition 4.4.6, $R^{(\alpha)} \rightsquigarrow_{\mathfrak{g}} T^*(G/H)$. So there is $h \in \mathfrak{h}, h \sim_G \alpha^\vee$ such that $(\mathbb{C}h, U) \rightsquigarrow_{\mathfrak{h}} \mathfrak{g}/\mathfrak{h}$, where U has a basis e_1, e_2 with $he_1 = 2e_1, he_2 = -2e_2$. The $\mathbb{C}h$-modules $\mathfrak{g}/\mathfrak{h}$ and $(\mathfrak{h}/\mathbb{C}h) \oplus U$ differ by a trivial summand whence (5.4). $\qquad\square$

Lemma 5.3.2. *Let $\mathfrak{h}$ be a nonzero Λ-essential subalgebra of $\mathfrak{g}$ such that $W(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{g})$.*

(1) *If $\mathfrak{g}$ is of types A, D, E, G, then $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$.*
(2) *If $\mathfrak{g}$ is of types B, C, F, then $\Lambda(\mathfrak{g}, \mathfrak{h}) = \operatorname{Span}_{\mathbb{Z}}(2\Delta(\mathfrak{g})^{min} \cup \Delta(\mathfrak{g})^{max})$ or $2\Lambda(\mathfrak{g})$.*

Proof. Recall that there is a basis of $\Lambda(\mathfrak{g}, \mathfrak{h})$ that is a root system with Weyl group $W(\mathfrak{g}, \mathfrak{h})$ (Proposition 3.2.11). Now the proof follows from Proposition 4.4.6. $\qquad\square$

Now we recall the definition of the Dynkin index ([D]). Let $\mathfrak{h}$ be a simple subalgebra of $\mathfrak{g}$. We fix an invariant non-degenerate symmetric bilinear form $K_{\mathfrak{g}}$ on $\mathfrak{g}$ such that $K_{\mathfrak{g}}(\alpha^\vee, \alpha^\vee) = 2$ for a root $\alpha \in \Delta(\mathfrak{g})$ of the maximal length. Analogously define a form $K_{\mathfrak{h}}$ on $\mathfrak{h}$. The *Dynkin index* of the embedding $\iota : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is, by definition, $K_{\mathfrak{g}}(\iota(x), \iota(x))/K_{\mathfrak{h}}(x, x)$ (the last fraction does not depend on the choice of $x \in \mathfrak{h}$ such that $K_{\mathfrak{h}}(x, x) \neq 0$). For brevity, we denote the Dynkin index of ι by $i(\mathfrak{h}, \mathfrak{g})$. It turns out that $i(\mathfrak{h}, \mathfrak{g})$ is a positive integer (see [D]).

For a simple Lie algebra $\mathfrak{h}$ let $k_{\mathfrak{h}}$ denote $\operatorname{tr}_{\mathfrak{h}}(\alpha^{\vee 2})$ for a long root $\alpha \in \Delta(\mathfrak{h})$. The numbers $k_{\mathfrak{h}}$ for all simple Lie algebras are given in Table 5.3.

Table 5.3: $k_{\mathfrak{h}}$.

$\mathfrak{h}$	A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$k_{\mathfrak{h}}$	$4l + 4$	$8l - 4$	$4l + 4$	$8l - 8$	48	72	120	36	16

Lemma 5.3.3. *Let $\mathfrak{h}$ be a nonzero Λ-essential subalgebra of $\mathfrak{g}$, $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$ the decomposition into the direct sum of simple ideals and $i_j := i(\mathfrak{h}_j, \mathfrak{g}), j = \overline{1, k}$.*

(1) *Suppose $\mathfrak{g}$ is of types A, B, D, E, F, $\operatorname{rk} \mathfrak{g} > 2$, $W(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{g})$ and $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$ (the last condition is essential only for $\mathfrak{g} \cong \mathfrak{so}_{2l+1}, F_4$). Then $\mathfrak{h}$ is semisimple and there are positive integers $a_j, j = \overline{1, k}$ such that*

(5.5)
$$\sum_{j=1}^{k} a_j i_j = 4,$$
$$\sum_{j=1}^{k} a_j k_{\mathfrak{h}_j} = 2k_{\mathfrak{g}} - 16.$$

(2) *Suppose $\mathfrak{g}$ is of type $C_l, l > 2, F_4$, $\mathfrak{h}$ is a Λ- essential subalgebra of $\mathfrak{g}$. Then $\Lambda(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}]) \neq \Lambda(\mathfrak{g})$. In other words, $\mathfrak{h}$ contains a nonzero Λ-essential semisimple ideal. Suppose, in addition, that $\mathfrak{h}$ is semisimple. Then there are nonnegative integers $a_j, j = \overline{1, k}$ such*

that

(5.6)
$$\sum_{j=1}^{k} a_j i_j = 8,$$
$$\sum_{j=1}^{k} a_j k_{\mathfrak{h}_j} = 4k_{\mathfrak{g}} - 16.$$

Further, if any proper ideal of $\mathfrak{h}$ is not Λ-essential, then $a_j > 0$ for any j and there is a subalgebra $\mathfrak{s} \subset \mathfrak{h}, \mathfrak{s} \sim_G \mathfrak{so}_3 \subset \mathfrak{sl}_3 \cong \mathfrak{g}^{(\alpha_1,\alpha_2)}, \alpha_1, \alpha_2 \in \Delta(\mathfrak{g})^{min}$, such that $\mathfrak{s}$ is not contained in a proper ideal of $\mathfrak{h}$.

(3) *Suppose $\mathfrak{g}$ is of types $A, C - F$. Then there are $h \in \mathfrak{h}$ satisfying (5.4) and a subalgebra $\mathfrak{s} \subset \mathfrak{h}, \mathfrak{s} \cong \mathfrak{sl}_2$, such that $h \sim_G \alpha^\vee, \alpha \in \Delta(\mathfrak{g})^{min}, h \in \mathfrak{s}$ and $\mathfrak{s}$ is not contained in a proper ideal of $[\mathfrak{h}, \mathfrak{h}]$.*

Proof. From [Lo5], Theorem 5.1.2, it follows that $s_\alpha \in W(\mathfrak{g}, \mathfrak{h})$ for $\alpha \in \Delta(\mathfrak{g})^{min}$ provided $\mathfrak{g} = F_4, \mathfrak{sp}_{2l}$. When $\mathfrak{g} \neq \mathfrak{so}_{2l+1}$ there are $\alpha_1, \alpha_2 \in \Delta(\mathfrak{g})^{min}$ such that $\mathfrak{g}^{(\alpha_1,\alpha_2)} \cong \mathfrak{sl}_3$. When $\mathfrak{g} = \mathfrak{so}_{2l+1}, l > 2, F_4$ there are $\alpha_1, \alpha_2 \in \Delta(\mathfrak{g})$ with $\mathfrak{g}^{(\alpha_1,\alpha_2)} \cong \mathfrak{sl}_3$. Applying Proposition 4.4.9, we see that $\widetilde{R}^{(\alpha_1,\alpha_2)} \leadsto_\mathfrak{g} T^*(G/H)$. Let $\mathfrak{s}$ denote a subalgebra in $\mathfrak{h}$ such that $\mathfrak{s} \sim_G \mathfrak{so}_3 \subset \mathfrak{g}^{(\alpha_1,\alpha_2)}$ and U the 5-dimensional irreducible $\mathfrak{s}$-module. Then $(\mathfrak{s}, U) \leadsto_\mathfrak{h} \mathfrak{g}/\mathfrak{h}$. Denote by $\mathfrak{h}^1$ the ideal in $\mathfrak{h}$ generated by $\mathfrak{s}$. Clearly, $(\mathfrak{s}, U) \leadsto_{\mathfrak{h}^1} \mathfrak{g}/\mathfrak{h}^1$. This implies assertion 3.

Since $(\mathfrak{s}, U) \leadsto_{\mathfrak{h}^1} \mathfrak{g}/\mathfrak{h}^1$, we have $\widetilde{R}^{(\alpha_1,\alpha_2)} \leadsto_\mathfrak{g} T^*(G/H^1)$. Set $a_j := \iota(\mathfrak{s}_j, \mathfrak{h}_j)$, where $\mathfrak{s}_j$ denotes the projection of $\mathfrak{s}$ to $\mathfrak{h}_j$. Thanks to Proposition 4.4.9, $\Lambda(\mathfrak{g}, \mathfrak{h}^1)$ equals $2\Lambda(\mathfrak{g})$ in assertion 1, and is contained $\text{Span}_\mathbb{Z}(2\Delta(\mathfrak{g})^{min} \cup \Delta(\mathfrak{g})^{max})$ in assertion 2. Hence if $\Lambda(\mathfrak{g}, \mathfrak{h}) = \text{Span}_\mathbb{Z}(2\Delta(\mathfrak{g})^{min} \cup \Delta(\mathfrak{g})^{max})$, we get $\mathfrak{h}^1 = \mathfrak{h}$, whence $a_j > 0$ for all $\mathfrak{g}$. In assertion 1 the equality $i(\mathfrak{s}, \mathfrak{g}) = 4$ holds, and in assertion 2 we have $i(\mathfrak{s}, \mathfrak{g}) = 8$. The first equalities in (5.5), (5.6) follow from the additivity property [Lo5], (5.1), of the Dynkin index proved in [D]. The $\mathfrak{s}$-modules $\mathfrak{g}/\mathfrak{h}$, $\mathfrak{h}/\mathfrak{s} \oplus U$ differ by a trivial summand. One gets the second equalities in (5.5), (5.6) by computing the traces of h^2, where h is a coroot in $\mathfrak{s} \cong \mathfrak{so}_3$, on these modules. $\square$

Now we prove some statements concerning reductive subalgebras in classical Lie algebras containing an element conjugate to $\alpha^\vee, \alpha \in \Delta(\mathfrak{g})$.

Proposition 5.3.4. *Let $\mathfrak{g}$ be a classical Lie algebra and $\mathfrak{h}$ a reductive subalgebra $\mathfrak{g}$ such that there is $h \in \mathfrak{h}$ such that $h \sim_G \alpha^\vee$ for $\alpha \in \Delta(\mathfrak{g})$ and h is not contained in a proper ideal of $\mathfrak{h}$.*

(1) *If $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{h}$ is semisimple, then $\mathfrak{h} = \mathfrak{sl}_k, \mathfrak{so}_k, \mathfrak{sp}_k$.*

(2) *If $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\alpha \in \Delta(\mathfrak{g})^{max}$, then $\mathfrak{h} = \mathfrak{so}_k, \mathfrak{gl}_k^{diag}, \mathfrak{spin}_8$.*

(3) *Suppose $\mathfrak{g} \cong \mathfrak{so}_{2n}$, $\mathfrak{h}$ is semisimple, and h is included into an $\mathfrak{sl}_2$-triple in $\mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{so}_k, \mathfrak{so}_k \oplus \mathfrak{so}_l, \mathfrak{sl}_k^{diag}, \mathfrak{sp}_k^{diag}, \mathfrak{so}_k^{diag}, \mathfrak{spin}_7, G_2, \mathfrak{spin}_8$.*

(4) *Suppose $\mathfrak{g} = \mathfrak{sp}_{2n}$, $\mathfrak{h}$ is semisimple, and $\alpha \in \Delta(\mathfrak{g})^{min}$. Then $\mathfrak{h} = \mathfrak{sp}_{2k}, \mathfrak{sp}_{2k} \oplus \mathfrak{sp}_{2l}, \mathfrak{sl}_k^{diag}, \mathfrak{so}_k^{diag}, \mathfrak{sp}_k^{diag}$.*

Proof. Let V denote the tautological $\mathfrak{g}$-module.

Step 1. Here we describe all semisimple subalgebras $\mathfrak{h} \subset \mathfrak{gl}(V)$ containing $h \in \mathfrak{gl}(V)$ such that:

(a) h is semisimple and its eigenvalues are ± 1, each of multiplicity 1, and 0 of multiplicity $\dim V - 2$.

(b) h is not contained in a proper ideal of $\mathfrak{h}$.

Since $\operatorname{tr}_U \xi = 0$ for any $\mathfrak{h}$-module U and $\xi \in \mathfrak{h}$, we see that $V/V^{\mathfrak{h}}$ is an irreducible $\mathfrak{h}$-module. All irreducible linear algebras $\mathfrak{h}$ containing such h where described in Proposition 8 from [Lo3]. These are $\mathfrak{sl}(V/V^{\mathfrak{h}}), \mathfrak{so}(V/V^{\mathfrak{h}})$ and $\mathfrak{sp}(V/V^{\mathfrak{h}})$.

Step 2. Here we describe all reductive subalgebras $\mathfrak{h} \subset \mathfrak{so}(V)$ containing h satisfying (a),(b). If $\mathfrak{h}$ is semisimple, then $\mathfrak{h} = \mathfrak{so}_k$ by step 1. Suppose $\mathfrak{h}$ is not semisimple. In this case $V/V^{\mathfrak{h}}$ is reducible. Analogously to step 1, there is no proper orthogonal submodule in $V/V^{\mathfrak{h}}$. Therefore there is an irreducible $\mathfrak{h}$-module V_0 such that $V/V^{\mathfrak{h}} = V_0 \oplus V_0^*$. We may assume that h acts on V_0 as $diag(1, 0, \ldots, 0)$. By [V], Proposition 2, $\mathfrak{h} = \mathfrak{gl}_k$.

Step 3. Here we classify all irreducible subalgebras $\mathfrak{h} \subset \mathfrak{gl}(V)$ such that the $\mathfrak{h}$-module V is self-dual and there is $h \in \mathfrak{h}$ that satisfies (b) and

(a′) h is semisimple and its eigenvalues are ± 1 of multiplicity 2 each and 0 of multiplicity $\dim V - 4$.

Choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h}$ containing h. We may assume that the positive root system $\Delta(\mathfrak{h})_+$ is chosen in such a way that $\langle \Delta(\mathfrak{h})_+, h \rangle \geqslant 0$. Let $\lambda_1, \ldots, \lambda_k$ be all different dominant weights of the $\mathfrak{h}$-module V, where λ_1 is the highest weight. Let us check that $k = 1$. Assume the converse. We note that $\langle \lambda_i, h \rangle > 0$ for all i. Indeed, being a dominant coweight, h is the sum of simple coroots with positive coefficients. Therefore $k = 2$ and both λ_1, λ_2 have multiplicity 1. Besides, as V is self-dual, $W(\mathfrak{h})\lambda_i = -W(\mathfrak{h})\lambda_i, i = 1, 2$. For all $\nu \in W(\mathfrak{h})\lambda_i, \nu \neq \pm\lambda_i$, we get $\langle \nu, h \rangle = 0$. It was shown in [V], Lemma 2, that $\mathfrak{h} = \mathfrak{so}_n, \mathfrak{sp}_n$ and the weight λ_i is proportional (up to an automorphism for $\mathfrak{h} = \mathfrak{so}_8$) to the highest weight of the tautological $\mathfrak{h}$-module. But λ_1 and λ_2 are not proportional, for $\langle \lambda_i, h \rangle = 1$. Thus $\mathfrak{h} = \mathfrak{so}_8$ and , up to an automorphism, $\lambda_1 = l\pi_1$, where $l > 1$. In this case $(l-2)\pi_1 + \pi_2$ is a weight of V. Contradiction.

So the highest weight is the only nonzero dominant weight of the $\mathfrak{h}$-module V.

At first, let us consider the case $V^{\mathfrak{t}} \neq 0$. In this case $\mathfrak{h}$ is simple and λ is the maximal short root.

Suppose $\mathfrak{h}$ is of types A, D, E. In this case $V = \mathfrak{h}$. There are exactly two positive roots having a nonzero pairing with h. These are the maximal root δ and another root, say, β. Clearly, β is a simple root and any root greater than β is maximal. Thus $\mathfrak{h} = A_2$.

If $\mathfrak{h} \cong \mathfrak{so}_{2l+1}, l > 1$, then $\mathfrak{h}$ contains a required element h.

If $\mathfrak{h} \cong \mathfrak{sp}_{2l}, l > 2, F_4$, then $\Delta(\mathfrak{h})^{min}$ is the root system D_l, and, by above, there is no $h \in \mathfrak{t}$ with required properties.

Finally, let $\mathfrak{h} = G_2$. In this case $\Delta(\mathfrak{h})^{min} = A_2$ and h exists.

Now consider the case $V^{\mathfrak{t}} = 0$. In this case λ_1 is minuscule, that is, $\langle \lambda_1, \delta^\vee \rangle = 1$, where $\delta^\vee$ denotes the maximal coroot. It follows that λ_1 is a fundamental weight, π_m. There is a unique weight less than λ_1 w.r.t. the natural order on the set of weights, namely, $\lambda_1 - \alpha_m$. This observation makes possible to find the system of linear equations for h. This system has a solution only in the following cases:

1) $\mathfrak{h} = \mathfrak{so}_{2n}, \mathfrak{sp}_{2n}$, V is the tautological $\mathfrak{h}$-module (or a half-spinor module for $\mathfrak{h} = \mathfrak{so}_8$).

2) $\mathfrak{h} = \mathfrak{so}_7, \lambda_1 = \pi_3$.

Step 4. Complete the proof of the proposition. Assertions 1 and 2 were proved on steps 1 and 2, respectively. Assertions 3 and 4 in the case when the $\mathfrak{h}$-module $V/V^{\mathfrak{h}}$ is reducible also follow from steps 1,2. If $V/V^{\mathfrak{h}}$ is irreducible, the image of $\mathfrak{h}$ in $\mathfrak{gl}(V/V^{\mathfrak{h}})$ is one of the subalgebras found on step 3. All of them except of $\operatorname{ad}(\mathfrak{sl}_3)$ fulfill the condition that h is included into an $\mathfrak{sl}_2$-triple in $\mathfrak{h}$. $\square$

Proof of Theorem 5.1.2. Throughout the proof $\mathfrak{h}$ denotes a Λ-essential subalgebra of $\mathfrak{g}$. Let H denote the connected subgroup of G with Lie algebra $\mathfrak{h}$ and $\widetilde{H}$ denote the inverse image of $Z(N_G(H)/H)$ in $N_G(H)$.

The case $\mathfrak{g} \cong \mathfrak{sl}_n$. At first, suppose $W(\mathfrak{g}, \mathfrak{h}) \neq W(\mathfrak{g})$. Let us check that $\Lambda(\mathfrak{g}, \mathfrak{h}) = \Lambda$, where $\Lambda := \operatorname{Span}_{\mathbb{Z}}(\alpha \in \Delta(\mathfrak{g})|s_\alpha \in W(\mathfrak{g}, \mathfrak{h}))$. By Proposition 3.2.11, there is an inclusion $\Lambda(\mathfrak{g}, \mathfrak{h}) \subset \Lambda$. To prove the inverse inclusion we need to check that $\alpha \in \Lambda(\mathfrak{g}, \mathfrak{h})$ for all $\alpha \in \Delta(\mathfrak{g})$ with $s_\alpha \in W(\mathfrak{g}, \mathfrak{h})$. Considering case by case possible groups $W(\mathfrak{g}, \mathfrak{h})$ ([Lo5], Theorem 5.1.2), we note that there is $\alpha_1 \in \Delta(\mathfrak{g})$ such that α, α_1 satisfy the assumptions of Corollary 4.4.8. Applying this corollary, we get $\Lambda(\mathfrak{g}, \mathfrak{h}) = \Lambda$.

Now suppose $W(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{g})$. By Lemma 5.3.2, $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$. Thanks to Lemma 5.3.1, $G\alpha^\vee \cap \mathfrak{h} \neq \varnothing$. By Lemma 5.3.3, $\mathfrak{h}$ is semisimple. Proposition 5.3.4 implies $\mathfrak{h} = \mathfrak{sl}_k, \mathfrak{so}_k, \mathfrak{sp}_{2k}$.

If $\mathfrak{h} = \mathfrak{sl}_k$, then, since $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{t}, W(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{g})$, we have $k \leqslant \frac{n}{2}$. By (5.5), $k = \frac{n}{2} - 1$. To show that $\Lambda(\mathfrak{g}, \mathfrak{h}) = \Lambda(\mathfrak{g})$ it is enough to note $\Lambda(\mathfrak{g}, \mathfrak{h}) \supset \Lambda(\mathfrak{g}, \mathfrak{sl}_{n/2})$ (see Proposition 3.2.6).

Let $\mathfrak{h} = \mathfrak{sp}_{2k}$. Analogously to the previous paragraph, we get $k = \frac{n}{2} - 2$ and $\Lambda(\mathfrak{g}, \mathfrak{h}) \supset \Lambda(\mathfrak{g}, \mathfrak{sp}_{n-2}) = \Lambda(\mathfrak{g})$.

Finally, suppose $\mathfrak{h} = \mathfrak{so}_k$. By (5.5), $k = n$. From [Krä], Tabelle 1, it follows that $\mathfrak{X}_{\mathrm{SL}_n, \mathrm{SL}_n / \mathrm{SO}_n} = \operatorname{Span}_{\mathbb{Z}}(2\pi_1, \ldots, 2\pi_{n-1})$. Since $\mathfrak{X}_{\mathrm{SL}_n, \mathrm{SL}_n / \mathrm{SO}_n} \subset \Lambda(\mathfrak{g}, \mathfrak{h})$, we get $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$.

The case $\mathfrak{g} = \mathfrak{so}_{2n+1}, n \geqslant 2$. At first, we consider the case $W(\mathfrak{g}, \mathfrak{h}) \neq W(\mathfrak{g})$. By [Lo5], $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{sl}_n^{diag}, G_2(n = 4), \mathfrak{spin}_7, (n = 5)$. Since $\Lambda(\mathfrak{g})$ is generated by $\Delta(\mathfrak{g})^{min}$, it follows from Proposition 4.4.6 that there is $h \in \mathfrak{h}, h \sim_G \alpha^\vee$. By Proposition 5.3.4, $\mathfrak{h} = \mathfrak{gl}_n^{diag}$. By [Krä], Tabelle 1, $\mathfrak{X}_{\mathrm{SO}_{2n+1}, \mathrm{SO}_{2n+1} / \mathrm{GL}_n^{diag}} = \operatorname{Span}_{\mathbb{Z}}\{\varepsilon_i\}|_{i=\overline{1,n}}$. Assume that $G = \mathrm{SO}_{2n+1}$. Choose nonzero vectors $v \in (\mathbb{C}^{2n+1})^H, \omega \in (\bigwedge^2 \mathbb{C}^{2n+1})^H$. The spaces $(\bigwedge^{2i+1} \mathbb{C}^{2n+1})^H, (\bigwedge^{2i} \mathbb{C}^{2n+1})^H$ are 1-dimensional, for G/H is spherical. These spaces are generated by $v \wedge \omega^{\wedge i}, \omega^{\wedge i}$, respectively. The group $N_G(H)/H$ is isomorphic to $\mathbb{Z}_2$. The nontrivial element of this group acts on $(\bigwedge^i \mathbb{C}^{2n+1})^H$ by $(-1)^{2\{i/2\}n+[i/2]}$. To show that $\Lambda(\mathfrak{g}, \mathfrak{h})$ has the required form we use Proposition 5.2.1.

Now suppose that $W(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{g})$. From Lemma 5.3.1 it follows that there is an element $h \in \mathfrak{h}$ satisfying (5.4) and such that $h \sim_{\mathrm{SO}_{2n+1}} diag(2, -2, 0 \ldots, 0)$ whence $\operatorname{tr}_{\mathfrak{g}} h^2 = 16n - 8$. By assertion 2 of Proposition 5.3.4, h is contained in an ideal of $\mathfrak{h}$ of the form $\mathfrak{gl}_k^{diag}$ or $\mathfrak{so}_k$. So we have $\operatorname{tr}_{\mathfrak{h}} h^2 = 8(k-1)$ (for $\mathfrak{gl}_k^{diag}$) or $\operatorname{tr}_{\mathfrak{h}} h^2 = 8(k-2)$ (for $\mathfrak{so}_k$). Thus the ideal $\mathfrak{h}_1$ of $\mathfrak{h}$ generated by h coincides with $\mathfrak{so}_{n+1}$.

Let us show, at first, that $\Lambda(\mathfrak{so}_{2n+1}, \mathfrak{so}_{n+1})$ has the form indicated in Table 5.1. By Lemma 5.3.3, $\Lambda(\mathfrak{so}_{2n+1}, \mathfrak{so}_{n+1}) \neq 2\Lambda(\mathfrak{g})$, for (5.5) is rewritten in the form $4(4n-4) = 2(8n-4) - 16$.

Suppose $G = \mathrm{SO}_{2n+1}$. By Proposition 5.2.1, $\Lambda(\mathfrak{g}, \mathfrak{h}) = \mathfrak{X}_{G, G/\widetilde{H}}$. Now it is enough to show that $L_{0\,G, G/\widetilde{H}} \neq \{1\}$. As Knop proved in [K1], Korollar 8.2, $L_{0\,G, G/\widetilde{H}}$ is the stabilizer in general position for the action $G : T^*(G/\widetilde{H})$. So it remains to show that the stabilizer in general position for the action $\widetilde{H} : \mathfrak{g}/\mathfrak{h}$ is nontrivial. This action coincides with the action of $\emptyset(n+1)$ on $(\mathbb{C}^{n+1})^{\oplus n}$. The stabilizer in general position is isomorphic to $\mathbb{Z}_2$.

Now suppose $\mathfrak{h} \neq \mathfrak{h}_1 = \mathfrak{so}_{n+1}$. By above, $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$. Assertion 1 of Lemma 5.3.3 implies that $\mathfrak{h}$ is semisimple. By Lemma 5.3.1, there is $h \in \mathfrak{h}$ such that and

$$(5.7) \qquad\qquad k_{\mathfrak{g}} = \operatorname{tr}_{\mathfrak{g}} h^2 = 2 \operatorname{tr}_{\mathfrak{h}} h^2 + 8$$

and $h \sim_{\mathrm{SO}_{2n+1}} diag(1, 1, -1, -1, 0 \ldots, 0)$. As we checked in the proof of the inequality $\Lambda(\mathfrak{g}, \mathfrak{h}_1) \neq 2\Lambda(\mathfrak{g})$, $h \notin \mathfrak{h}_1$. Further, $h \notin \mathfrak{h}_1^\perp$. Otherwise, $\operatorname{tr}_{\mathfrak{h}} h^2 = \operatorname{tr}_{\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})} h^2$ and (5.7) does not hold.

Let us check that if $\mathfrak{h} \neq \widehat{\mathfrak{h}} := \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}_1) = \mathfrak{so}_n \oplus \mathfrak{so}_{n+1}$, then $\operatorname{tr}_{\mathfrak{h}} h^2 < \operatorname{tr}_{\widetilde{\mathfrak{h}}} h^2$. Otherwise, h acts trivially on $\widehat{\mathfrak{h}}/\mathfrak{h}$, for $2 \operatorname{tr}_{\widehat{\mathfrak{h}}_1} h^2 = k_{\mathfrak{g}} - 8$. So $\mathfrak{h}$ and $\widehat{\mathfrak{h}}$ have a common ideal containing h. Since $h \notin \mathfrak{h}_1$, this is impossible whence $\mathfrak{h} = \mathfrak{so}_{n+1} \oplus \mathfrak{so}_n$. Let us check that $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$. Indeed, $N_G(H) = \operatorname{SL}_{2n+1} \cap (\varnothing_n \times \varnothing_{n+1})$, $\Lambda(\mathfrak{g}, \mathfrak{h}) = \mathfrak{X}_{G, G/N_G(H)}$, and the last lattice is easily extracted from [Krä], Tabelle 1.

The case $\mathfrak{g} = \mathfrak{sp}_{2n}, n > 2$.

At first, we determine all nonzero Λ-essential subalgebras $\mathfrak{h} \subset \mathfrak{g}$ whose proper ideals are not Λ-essential. Thanks to assertion 2 of Lemma 5.3.3, we see that $\mathfrak{h}$ is semisimple. Applying Theorem 5.1.2 from [Lo5], we see that $W(\mathfrak{g}, \mathfrak{h})$ contains s_α for any $\alpha \in \Delta(\mathfrak{g})^{min}$. By assertion 2 of Lemma 5.3.1, there is a subalgebra $\mathfrak{s} \subset \mathfrak{h}$ not contained in a proper ideal of $\mathfrak{h}$ such that $\mathfrak{s} \sim_{\operatorname{Sp}_{2n}} \mathfrak{so}_3^{diag}$. Taking into account assertion 3 of Proposition 5.3.4, we see that $\mathfrak{h} = \mathfrak{sp}_{2k}, \mathfrak{sl}_k^{diag}, \mathfrak{so}_k^{diag}$. Only $\mathfrak{sl}_n^{diag}, \mathfrak{sp}_{n-2}$ satisfy (5.6).

Let us show that $\Lambda(\mathfrak{sp}_{4m+2}, \mathfrak{sp}_{2m}) = \Lambda(\mathfrak{g})$. Set $G = \operatorname{Ad}(\operatorname{Sp}_{4m+2})$. The center $N_G(H)/H$ is $\{1\}$. By Proposition 5.2.1, $\Lambda(\mathfrak{g}, \mathfrak{h}) = \mathfrak{X}_{G, G/H}$. So it remains to prove that the s.g.p. for the action $H : \mathfrak{g}/\mathfrak{h}$ is trivial. The last action coincides with the natural action of Sp_{2m} on $(\mathbb{C}^{2m})^{\oplus 2m+2}$. But the s.g.p. is trivial already for $\operatorname{Sp}_{2m} : (\mathbb{C}^{2m})^{\oplus 2m}$.

Suppose $\mathfrak{h} = \mathfrak{sl}_n^{diag}$. Let us check that $\Lambda(\mathfrak{g}, \mathfrak{h}) = \operatorname{Span}_{\mathbb{Z}}(2\Delta(\mathfrak{g})^{min} \cup \Delta(\mathfrak{g})^{max})$. At first, we show that $\Lambda(\mathfrak{sp}_{2n}, \mathfrak{gl}_n^{diag}) = 2\Lambda(\mathfrak{g})$. Indeed the subalgebra $\mathfrak{gl}_n^{diag} \subset \mathfrak{sp}_{2n}$ is spherical, and the subgroup $\operatorname{GL}_n^{diag} \subset \operatorname{Sp}_{2n}$ is of index 2 in its normalizer. By Proposition 5.2.1, $\Lambda(\mathfrak{sp}_{2n}, \mathfrak{gl}_n^{diag})$ is also of index 2 in $\mathfrak{X}_{\operatorname{Sp}_{2n}, \operatorname{Sp}_{2n}/\operatorname{GL}_n^{diag}}$. By [Krä], the last lattice equals $\operatorname{Span}_{\mathbb{Z}}(2\pi_1, \ldots, 2\pi_n)$ whence the equality for $\Lambda(\mathfrak{sp}_{2n}, \mathfrak{gl}_n^{diag})$.

By assertion 2 of Lemma 5.3.3, $\Lambda(\mathfrak{g}, \mathfrak{h}) \subset \operatorname{Span}_{\mathbb{Z}}(2\Delta(\mathfrak{g})^{min} \cup \Delta(\mathfrak{g})^{max})$. The equality will follow if we check that $\Lambda(\mathfrak{g}, \mathfrak{h}) \neq 2\Lambda(\mathfrak{g})$. Assume the converse. By Lemma 5.3.1, there is $h_0 \in \mathfrak{h}, h_0 \sim_{\operatorname{Sp}_{2n}} diag(1, -1, 0, \ldots, 0)$, which is absurd.

It remains to prove that $\mathfrak{gl}_n \subset \mathfrak{sp}_{2n}$ is the only subalgebra $\mathfrak{h}$ satisfying $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$. Indeed, let $\mathfrak{h}$ be such a subalgebra. By assertion 2 of Lemma 5.3.3, $\Lambda(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}]) \neq \Lambda(\mathfrak{g})$. Thus $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{sl}_n$.

The case $\mathfrak{g} = \mathfrak{so}_{2n}, n \geqslant 8$. Let $\mathfrak{h}$ be a Λ-essential subalgebra of $\mathfrak{g}$. By assertion of 1 Lemma 5.3.3, $\mathfrak{h}$ is semisimple. According to Lemma 5.3.1, $\mathfrak{h}$ contains an element h satisfying (5.4) such that $h \sim_{\operatorname{SO}_{2n}} diag(1, 1, -1, -1, 0 \ldots, 0)$. By assertion 3 of Lemma 5.3.3, we may assume that some multiple of h can be included into an $\mathfrak{sl}_2$-triple not contained in a proper ideal of $\mathfrak{h}$. Assertion 4 of Proposition 5.3.4 implies that $\mathfrak{h} = \mathfrak{so}_k, \mathfrak{so}_k \oplus \mathfrak{so}_l, \mathfrak{sl}_k^{diag}, \mathfrak{sp}_k^{diag}, \mathfrak{so}_k^{diag}, \mathfrak{spin}_7$, G_2. We note that $\mathfrak{h} \neq \mathfrak{sl}_n, \mathfrak{so}_k, k > n$, for $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{t}$.

Firstly, consider the case $\mathfrak{h} = \mathfrak{so}_k \oplus \mathfrak{so}_l$, where $k, l \leqslant n$. Here the projection of h to both ideals $\mathfrak{so}_k, \mathfrak{so}_l$ is conjugate to $diag(1, -1, 0, \ldots, 0)$. (5.4) holds iff $k = l = n$. Let us check that indeed $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$. We may assume $G = \operatorname{SO}_{2n}$. The homogeneous space G/H is spherical and $\#N_G(H)/H = 2$. Thus $\Lambda(\mathfrak{g}, \mathfrak{h}) = \mathfrak{X}_{G, G/N_G(H)}$ is of index 2 in $\mathfrak{X}_{G, G/H}$. The required equality follows easily from Tabelle 1 of [Krä].

Among the remaining subalgebras $\mathfrak{h}$ only $\mathfrak{sl}_{n-2} \subset \mathfrak{so}_{2n}$, $\mathfrak{spin}_7 \subset \mathfrak{so}_{12}, G_2 \subset \mathfrak{so}_{10}$ satisfy (5.5). Let us check that in these cases $\Lambda(\mathfrak{g}, \mathfrak{h}) = \Lambda(\mathfrak{g})$. For $\mathfrak{sl}_{n-2}$ this stems from the inclusion $\mathfrak{sl}_{n-2} \subset \mathfrak{sl}_{n-1}$. In the other cases take $\operatorname{Ad}(\operatorname{SO}_{2m})$ for G. Note that $N_G(H)/H \cong \operatorname{Ad}(\operatorname{SO}_k)$ for $k = 3, 4$. By Proposition 5.2.1, $\Lambda(\mathfrak{g}, \mathfrak{h}) = \mathfrak{X}_{G, G/H}$. To prove the equality $\mathfrak{X}_{G, G/H} = \Lambda(\mathfrak{g})$ it is enough to check that the s.g.p. for the action $H : \mathfrak{g}/\mathfrak{h}$ is trivial. This follows from the classification of Popov, [Po1].

The case $\mathfrak{g} = E_6$. By Lemma 5.3.3, $\mathfrak{h}$ is semisimple. Let $\mathfrak{h}_j, i_j, a_j, j = \overline{1, k}$, be such as in assertion 1 of Lemma 5.3.3. We reorder $\mathfrak{h}_j$ in such a way that $k_{\mathfrak{h}_1} \geqslant k_{\mathfrak{h}_2} \geqslant \ldots \geqslant k_{\mathfrak{h}_k}$. Since

$\mathfrak{a}(\mathfrak{g},\mathfrak{h}) = \mathfrak{t}$, we have $\mathfrak{h}_i \neq D_5, A_5, B_4, F_4$. (5.5) can be rewritten as

(5.8)
$$\sum_{j=1}^{k} a_j i_j = 4,$$
$$\sum_{j=1}^{k} a_j k_{\mathfrak{h}_j} = 80.$$

Thus $k_{\mathfrak{h}_1} \geqslant 20$. It follows that $\mathfrak{h}_1$ is one of the subalgebras $A_4, B_3, C_4, D_4 \subset E_6$.

If $\mathfrak{h}_1 = D_4$, then $\mathfrak{h} = \mathfrak{h}_1$, for $\mathfrak{n}_\mathfrak{g}(\mathfrak{h}_1)/\mathfrak{h}_1$ is commutative. This contradicts (5.8). The subalgebra $\mathfrak{h}_1 = B_3$ is embedded into D_4 and $\mathfrak{h}_1 = [\mathfrak{n}_\mathfrak{g}(\mathfrak{h}_1), \mathfrak{n}_\mathfrak{g}(\mathfrak{h}_1)]$ so in this case $\Lambda(\mathfrak{g},\mathfrak{h}) = \Lambda(\mathfrak{g})$ too.

Consider the case $\mathfrak{h}_1 = A_4$. It is easy to see that $\mathfrak{n}_\mathfrak{g}(\mathfrak{h}_1)/\mathfrak{h}_1 \cong \mathbb{C} \times \mathfrak{sl}_2$. If $\mathfrak{h} \neq \mathfrak{h}_1$, then $\mathfrak{h} = A_4 \times A_1$. However in this case (5.8) has no positive solutions. So $\mathfrak{h} = A_4$. Take $\mathrm{Ad}(E_6)$ for G. The subalgebra $\mathfrak{h}$ is included into D_5. So $N_G(\mathfrak{h})$ acts on $\mathfrak{h}$ as $\mathrm{Aut}(\mathfrak{h})$. Clearly, $N_G(H)^\circ$ is a Levi subgroup of G. From this we deduce that $N_G(H)$ has exactly two connected components. Choose $\sigma \in N_G(H) \setminus N_G(H)^\circ$. Let Z, F denote the center and the commutant of $(N_G(H)/H)^\circ$, respectively. The element σ acts on $\mathfrak{z}$ by -1. Therefore the image of $Z(N_G(H)/H)$ under the projection $(N_G(H)/H)^\circ \to (N_G(H)/H)^\circ/F$ is isomorphic to $\mathbb{Z}_2$. On the other hand, the center of F is of order at most 2. So $\#Z(N_G(H)/H) \leqslant 4$. By Proposition 5.2.1, $\mathfrak{X}_{G,G/\widetilde{H}} = \Lambda(\mathfrak{g},\mathfrak{h})$. If $\Lambda(\mathfrak{g},\mathfrak{h}) \neq \Lambda(\mathfrak{g})$, then $L_{0\,G,G/\widetilde{H}} \cong \Lambda(\mathfrak{g})/2\Lambda(\mathfrak{g}) \cong \mathbb{Z}_2^6$. So the s.g.p. for the action $\widetilde{H} : \mathfrak{g}/\mathfrak{h}$ is isomorphic to $\mathbb{Z}_2^6$. Therefore the s.g.p. for the action $H : \mathfrak{g}/\mathfrak{h}$ is nontrivial. Clearly, $\mathfrak{g}/\mathfrak{h} \cong (\bigwedge^2 \mathbb{C}^5 \oplus \bigwedge^2 \mathbb{C}^{5*} \oplus \mathbb{C})^{\oplus 2} \oplus \mathbb{C}^5 \oplus \mathbb{C}^{5*}$. By [Po1], the s.g.p. for this action is trivial.

Finally, let us consider the case $\mathfrak{h} = C_4$. In this case the inequality $\mathfrak{X}_{G,G/H} \neq \Lambda(\mathfrak{g})$ stems from [Krä], Tabelle 1.

The case $\mathfrak{g} = E_7$. In this case, by Lemma 5.3.3, $\mathfrak{h}$ is semisimple. Define $\mathfrak{h}_j, i_j, a_j, j = \overline{1,k}$, analogously to the previous case. Since $\mathfrak{a}(\mathfrak{g},\mathfrak{h}) = \mathfrak{t}$, $\mathfrak{h}_i \neq E_6, D_6$. (5.5) is rewritten as

(5.9)
$$\sum_{j=1}^{k} a_j i_j = 4,$$
$$\sum_{j=1}^{k} a_j k_{\mathfrak{h}_j} = 128.$$

Thus $k_{\mathfrak{h}_1} \geqslant 32$. It follows that $\mathfrak{h}_1 = A_7, D_5, B_5, F_4$. If $\mathfrak{h}_1 = B_5, F_4$, then $k_{\mathfrak{h}_1} = 36$. Therefore $\mathfrak{h} \neq \mathfrak{h}_1, a_1 \leqslant 3$, and $k_{\mathfrak{h}_2} \geqslant 20$. One easily sees that this is impossible. If $\mathfrak{h}_1 = D_5$, then $k_{\mathfrak{h}_1} = 32$ whence $\mathfrak{h} = \mathfrak{h}_1$. But D_5 is included into B_5 whence $\Lambda(E_7, D_5) = \Lambda(\mathfrak{g})$. Finally, for $\mathfrak{h} = A_7$ the inequality $\Lambda(\mathfrak{g},\mathfrak{h}) \neq \Lambda(\mathfrak{g})$ follows from [Krä], Tabelle 1.

The case $\mathfrak{g} = E_8$. Again, $\mathfrak{h}$ is semisimple. Let $\mathfrak{h}_j, i_j, a_j, j = \overline{1,k}$ be such as in the case E_6. Note that $\mathfrak{h}_1 \neq E_7$. (5.5) is rewritten as

(5.10)
$$\sum_{j=1}^{k} a_j i_j = 4,$$
$$\sum_{j=1}^{k} a_j k_{\mathfrak{h}_j} = 224.$$

Therefore $k_{\mathfrak{h}_1} \geqslant 56$. Hence $\mathfrak{h}_1 = D_8$. The inequality $\Lambda(\mathfrak{g}, \mathfrak{h}) \neq \Lambda(\mathfrak{g})$ follows from [Krä], Tabelle 1.

The case $\mathfrak{g} = F_4$. At first, suppose that $\mathfrak{h}$ does not have nonzero Λ-essential ideals. By assertion 2 of Lemma 5.3.3, $\mathfrak{h}$ is semisimple. Let $a_j, i_j, k_{\mathfrak{h}_j}$ have the same meaning as in the case $\mathfrak{g} = E_6$. Since $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{t}$, we see that $\mathfrak{h} \neq B_4, D_4$. (5.6) can be rewritten as

$$\sum_{j=1}^{k} a_j i_j = 8,$$

(5.11)

$$\sum_{j=1}^{k} a_j k_{\mathfrak{h}_j} = 128.$$

It follows that $k_{\mathfrak{h}_1} \geqslant 16$. Thus $\mathfrak{h}_1 = A_3, C_3, B_3$ or G_2. If $\mathfrak{h}_1 = A_3$, then $k_{\mathfrak{h}_1} = 16$. Thus $a_1 = 8$ and $\mathfrak{h} = \mathfrak{h}_1$. However A_3 does not contain a subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2$ of index 8, contradiction with assertion 2 of Lemma 5.3.3.

Suppose $\mathfrak{h}_1 = B_3$. Since $k_{\mathfrak{h}_1} = 20$, we see that $\mathfrak{h} \neq \mathfrak{h}_1$. This contradicts $\mathfrak{h}_1 = [\mathfrak{n}_\mathfrak{g}(\mathfrak{h}_1), \mathfrak{n}_\mathfrak{g}(\mathfrak{h}_1)]$. So $\Lambda(F_4, B_3) = \Lambda(\mathfrak{g})$. If $\mathfrak{h}_1 = G_2$, then again $\mathfrak{h}_1 = [\mathfrak{n}_\mathfrak{g}(\mathfrak{h}_1), \mathfrak{n}_\mathfrak{g}(\mathfrak{h}_1)]$ whence $\mathfrak{h} = \mathfrak{h}_1$. However, G_2 is included into B_3, contradiction.

Finally, consider the case $\mathfrak{h}_1 = C_3$. Since $k_{\mathfrak{h}_1} = 16$, we have $\mathfrak{h} = \mathfrak{h}_1$. The H-modules $\mathfrak{g}/\mathfrak{h} \cong V(\pi_3)^{\oplus 2}$ are isomorphic. As Popov proved in [Po1], the s.g.p. for the action $\mathrm{Sp}(6) : V(\pi_3)^{\oplus 2}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $L_{0\,G,G/H} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ whence $\Lambda(\mathfrak{g}, \mathfrak{h}) \neq \Lambda(\mathfrak{g})$. By Theorem 5.1.2 from [Lo5], $W(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{g})$. Since the system (5.5) has no solution, we get $\Lambda(\mathfrak{g}, \mathfrak{h}) \neq 2\Lambda(\mathfrak{g})$.

Now let us determine all Λ-essential subalgebras $\mathfrak{h} \subset \mathfrak{g}$ that have the ideal C_3. In particular, $W(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{g})$. By assertion 1 of Lemma 5.3.3, $\mathfrak{h}$ is semisimple. The only possibility is $\mathfrak{h} = C_3 \times A_1$. The equality $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$ follows from [Krä], Tabelle 1.

The case $\mathfrak{g} = G_2$. By Lemma 5.3.1, $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$ and there is $h \in \mathfrak{h}, h \sim_G \alpha^\vee, \alpha \in \Delta(\mathfrak{g})^{min}$ such that

(5.12) $\mathrm{tr}_\mathfrak{h}\, h^2 = 4.$

Since $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{t}$, we have $\mathfrak{h} \neq A_2$. Note that $\mathfrak{h} \cap G\beta^\vee \neq \varnothing$ for $\beta \in \Delta(\mathfrak{g})^{max}$. Thus $\mathrm{rk}\, \mathfrak{h} = 2$ and we may assume that $\mathfrak{t} \subset \mathfrak{h}$. There are three (up to sign) elements in $\mathfrak{t}$ that are G-conjugate with $\alpha^\vee$. Considering them case by case, we see that if (5.12) holds, then $\mathfrak{h} = A_1 \times \widetilde{A}_1$. Here $\Lambda(\mathfrak{g}, \mathfrak{h}) = 2\Lambda(\mathfrak{g})$ stems from [Krä], Tabelle 1. $\square$

5.4. **Proof of Theorem 5.1.3.** At first, we check that $H^{\mathfrak{X}-sat} \subset N_G(H)$. By Lemma 3.2.4, $H^{\mathfrak{X}-sat}/\widehat{H} \subset Z(N_G(\widehat{H})/\widehat{H})$, where the subgroup $\widehat{H} \subset N_G(H)$ was defined in Subsection 5.1 before Theorem 5.1.3. It is obvious that $H \subset N_G(\widehat{H})$ whence the claim. Note also that $H_0 = H \cap H^{\mathfrak{X}-sat}$ is a normal subgroup in H.

Now let us show that $\mathfrak{X}_{G,G/H} = \mathfrak{X}_{G,G/H_0}$. By the definition of H_0, we have $\Lambda(\mathfrak{g}, \mathfrak{h}_0) = \Lambda(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{X}_{G,G/H} \subset \mathfrak{X}_{G,G/H_0}$. The inclusion $\mathfrak{X}_{G,G/H}/\Lambda(\mathfrak{g}, \mathfrak{h}) \hookrightarrow \mathfrak{X}_{G,G/H_0}/\Lambda(\mathfrak{g}, \mathfrak{h})$ corresponds to the epimorphism $\mathfrak{A}_{G,G/H_0} \twoheadrightarrow \mathfrak{A}_{G,G/H}$ (existing by Proposition 3.2.8). It remains to check that the kernel of the last homomorphism is trivial. By the uniqueness part of Proposition 3.2.8, the homomorphism $\mathfrak{A}_{G,G/H_0} \to \mathrm{Aut}^G(G/H) \cong N_G(H)/H$ coincides with $H^{\mathfrak{X}-sat}/H_0 \to N_G(H)/H$. The latter is injective, for $H_0 = H \cap H^{\mathfrak{X}-sat}$.

Let us show that $\mathfrak{h}_0 = [\mathfrak{h}^{\mathfrak{X}-sat}, \mathfrak{h}^{\mathfrak{X}-sat}]$. This will immediately yield $H_0^{\mathfrak{X}-sat} = H^{\mathfrak{X}-sat}$. Let us note that $[\mathfrak{h}^{\mathfrak{X}-sat}, \mathfrak{h}^{\mathfrak{X}-sat}] = \widehat{\mathfrak{h}} \subset \mathfrak{h}$ by the definition of $H^{\mathfrak{X}-sat}$. The required equality follows from the observation that $\widehat{\mathfrak{h}}$ is the maximal ideal of $\mathfrak{h}^{\mathfrak{X}-sat}$ such that $\mathfrak{a}(\mathfrak{g}, \widehat{\mathfrak{h}}) = \mathfrak{t}$.

Proceed to the proof of assertion 2. The only nontrivial claim here is the particular form of the duality for algebras $\mathfrak{h}$ from Table 5.1. The Frobenius reciprocity implies that $\mathfrak{X}_{G,G/H^\circ}$ is spanned by all λ with $V(\lambda^*)^\mathfrak{h} \neq \{0\}$ (as $\mathfrak{h}$ is reductive, the latter is equivalent to $V(\lambda)^\mathfrak{h} \neq \{0\}$). Besides, if $V(\lambda)^\mathfrak{h} \neq \{0\}$, then χ_λ coincides with the character by that $H^{\mathfrak{X}-sat}/H^\circ$ acts on $V(\lambda^*)^\mathfrak{h}$. Hence the claim on the restriction of χ_λ to $Z(G)/(Z(G) \cap H^\circ)$.

All subalgebras except NN2,4,7,11 are spherical and the lattices $\mathfrak{X}_{G,G/H^\circ}$ were computed in [Krä], Tabelle 1. In cases 2,4 the lattice $\mathfrak{X}_{G,G/H^\circ}$ coincides with $\mathfrak{X}(G)$. This easily follows from observations of the previous paragraph. In case 7 the lattice is extracted from tables in [Pa3]. Finally, in case 11 we have seen in the proof of Theorem 5.1.2 that $L_{0\,G,G/H^\circ} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $\mathfrak{X}_G/\Lambda(\mathfrak{g},\mathfrak{h}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have $\mathfrak{X}_{G,G/H^\circ} = \Lambda(\mathfrak{g},\mathfrak{h})$. In all cases in consideration $H^{\mathfrak{X}-sat}/H^\circ = Z(N_G(H^\circ))/H^\circ$. Finding a generator (a typical element in cases 2,3) is not difficult. The character χ_λ is computed by using the remarks of the previous paragraph.

Assertion 3 follows directly from the definition of the duality.

5.5. Finding distinguished components.
Below in this subsection $X = G/H$ and $\underline{X}$ is the distinguished component of $X^{L_{0\,G,X}}$. To find a point from $\underline{X}$ we use the following proposition (compare with [Lo5], Proposition 4.1.3).

Proposition 5.5.1. *We use the notation established in Subsection 5.1.*

(1) *Let $H_0 = H \cap H^{\mathfrak{X}-sat}$, $\widetilde{X} = G/H_0$, $\pi : \widetilde{X}_0 \to X$ be the natural morphism and $\underline{\widetilde{X}}$ the distinguished component of $\widetilde{X}^{L_{0\,G,X}}$. Then $\pi(\underline{\widetilde{X}}) \subset X$.*

(2) *Suppose $H \subset H^{\mathfrak{X}-sat}$, and let $\pi : G/H \twoheadrightarrow G/H^{\mathfrak{X}-sat}$ be the natural epimorphism and $\underline{X}'$ the distinguished component of $(G/H^{\mathfrak{X}-sat})^{L_{0\,G,G/H^{\mathfrak{X}-sat}}}$. Then $\pi^{-1}(\underline{X}') \subset \underline{X}$.*

(3) *Let $G = G_1 \times G_2, H = H_1 \times H_2$. Then $\underline{X}$ coincides with the product of the distinguished components of $(G_i/H_i)^{L_{0\,G_i,G_i/H_i}}$.*

(4) *Suppose $\mathfrak{g}$ is simple and $\mathfrak{h}$ is the subalgebra from Table 5.1 embedded into $\mathfrak{g}$ as indicated below. Then $eH^{\mathfrak{X}-sat}$ lies in the distinguished component of $(G/H^{\mathfrak{X}-sat})^{L_{0\,G,G/H^{\mathfrak{X}-sat}}}$.*

Let us describe the embeddings $\mathfrak{h} \to \mathfrak{g}$ in consideration. In cases 1,5,8-10,12-15 we embed $\mathfrak{h}$ into $\mathfrak{g}$ as the fixed-point subalgebra of a Weyl involution σ, i.e., an involutory automorphism of $\mathfrak{g}$ fixing $\mathfrak{t}$ and acting on $\mathfrak{t}$ by -1. An involution σ is defined uniquely up to T-conjugacy.

In cases 2,3 the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is such as in [Lo5], Subsection 4.4 .

In case 4 we embed $\mathfrak{h}$ into $\mathfrak{g}$ as the annihilator of the vectors $e_i + e_{2n+2-i}, i = \overline{1,n}$.

In case 6 we embed $\mathfrak{h}$ into $\mathfrak{g}$ as the stabilizer of the isotropic subspaces $U_\pm$ spanned by vectors of the form $x \pm i\iota(x)$, where $x \in \mathrm{Span}_{\mathbb{C}}(e_{2i}, i \leqslant i)$ and ι is an isometrical embedding $\mathrm{Span}_{\mathbb{C}}(e_{2i})$ into $\mathrm{Span}_{\mathbb{C}}(e_{2i-1}, i \leqslant n+1)$.

In cases 7,11 $\mathfrak{h} = \mathfrak{g}^{(\alpha_1,...,\alpha_n-1)}, \mathfrak{g}^{(\alpha_1,\alpha_2,\alpha_3)}$, respectively.

Proof of Proposition 5.5.1. To prove assertions 1-3 one argues exactly as in the proof of the analogous assertions of [Lo5],Proposition 4.1.3, (the group $L_{0\,\bullet,\bullet}^\circ$ there should by replaced with $L_{0\,\bullet,\bullet}$). Let us prove assertion 4.

Suppose that $\mathfrak{h} = \mathfrak{g}^\sigma$. Then $\mathfrak{h} \oplus \mathfrak{b} = \mathfrak{g}$, $\mathfrak{b} \cap \mathfrak{h}^\perp = \mathfrak{t}$. It follows that $B \cap N_G(\mathfrak{h}) \subset T$ whence $A_{G,G/N_G(\mathfrak{h})} \cong T/T \cap H$. The last equality is equivalent to $T \cap H = L_{0\,G,G/H^{\mathfrak{X}-sat}}$ (recall that in the cases in interest H is spherical whence $H^{\mathfrak{X}-sat} = N_G(\mathfrak{h})$). Since the B-orbit of $eH^{\mathfrak{X}-sat}$ is dense in $G/H^{\mathfrak{X}-sat}$, we see that eH is contained in the distinguished component.

In cases 2, 3 we get $L_{0\,G,G/H^{\mathfrak{X}-sat}} \cong \mathbb{C}^\times$ and the claim follows from [Lo5], Proposition 4.1.3.

Below we suppose $H = H^{\mathfrak{X}-sat}$ and set $L_0 := L_{0\,G,G/H^{\mathfrak{X}-sat}}$. According to [Lo5], Proposition 4.3.2, we only need to check that $L_0 \subset H$, $\dim G - \dim N_G(L_0) = 2(\dim H - \dim N_H(L_0))$, and $N_G(L_0) = N_G(L_0)^\circ N_H(L_0)$.

In case 4 $L_0 = \{diag(-1,\ldots,-1,1,-1,\ldots,-1)\}$ (here and in the next case we assume $G = \mathrm{SO}(2n+1)$). Therefore $L_0 \subset H$. Further, $N_G(L_0) \cong \mathrm{S}(\emptyset_{2n} \times \emptyset_1)$, $N_H(L_0) \cong \mathrm{S}(\emptyset_n \times \emptyset_1)$ whence the two remaining equalities.

In case 6 $L_0 = \{diag(\pm 1, \mp 1, \ldots, -1, 1, -1, \ldots, \mp 1, \pm 1)\}$. The nontrivial element of L_0 transposes $\mathrm{GL}(n)$-stable isotropic subspaces whence $L_0 \subset H$. Further, $N_G(L_0) \cong \mathrm{S}(\emptyset_{n+1} \times \emptyset_n)$. On the other hand, embed $\emptyset_n \cong \emptyset(\mathrm{Span}_{\mathbb{C}}(e_{2i}))$ into $\emptyset_{2n+1}$ so that $\emptyset_n$ acts on $\mathrm{Span}_{\mathbb{C}}(e_{2i})$ in the initial way, on $\mathrm{im}\,\iota$ via ι and on the orthogonal complement of the sum of these two spaces trivially. Then $\emptyset(n) = N_{H^\circ}(L_0)$ and the required equalities follow.

In case 7 $L_0 = diag(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_n, \ldots, \varepsilon_1)$, where $\varepsilon_i \in \{\pm 1\}$ (we consider $G = \mathrm{Sp}_{2n}$). Clearly, $L_0 \subset H$. Further, $N_G(L_0)^\circ \cong \mathrm{SL}_2^n$, while $N_H(L_0)^\circ$ is a maximal torus of H. The groups $N_G(L_0)/N_G(L_0)^\circ$, $N_H(L_0)/N_H(L_0)^\circ$ acts on L_0 as the symmetric group on n elements.

In case 11 the group L_0 is generated by $\exp(\pi i \alpha_1^\vee), \exp(\pi i \alpha_2^\vee)$. The equalities $\mathfrak{g}^{L_0} = D_4, \mathfrak{h}^{L_0} \cong \mathfrak{sl}_2^3$ hold. Any component of $N_G(L_0)$ contains an element of $N_G(T)$. It remains to note that the Weyl group $W(D_4)$ is normal in $W(\mathfrak{g})$ and $W(\mathfrak{h})W(D_4) = W(\mathfrak{g})$. $\square$

6. Algorithm

Here we provide an algorithm computing the weight lattice $\mathfrak{X}_{G,X}$, where X is a homogeneous space or an affine homogeneous vector bundle.

Case 1. $X = G/H$, where H is a reductive subgroup of G and $\mathrm{rk}_G(G/H) = \mathrm{rk}(G)$. The lattice $\mathfrak{X}_{G,G/H}$ is computed as indicated in Theorem 5.1.3. To compute the distinguished component of $(G/H)^{L_{0\,G,G/H}}$ we use Proposition 5.5.1.

Case 2. Suppose $X = G/H$, where H is a reductive subgroup of G. Using Theorem 1.3, [Lo4], we compute $\mathfrak{a}(\mathfrak{g}, \mathfrak{h})$. Then, applying Proposition 4.1.3 from [Lo5], we compute a point in the distinguished component of $(G/H)^{L_0}$, $L_0 := L_{0\,G,G/H}^\circ$. We may assume that eH lies in that distinguished component. Then applying [Lo5], Proposition 4.1.2, we determine the whole distinguished component, which is an affine homogeneous $\underline{G}/\underline{H}$ with $\mathrm{rk}_{\underline{G}}(\underline{G}/\underline{H}) = \mathrm{rk}\,\underline{G}$, $\underline{G} := N_G(L_0)^\circ/L_0$, $\underline{H} := (H \cap N_G(L_0)^\circ)/L_0$. We know that $\mathfrak{X}_{G,G/H} = \mathfrak{X}_{\underline{G},\underline{G}/\underline{H}}$ (see Proposition 3.3.5). The last lattice is computed as in case 1. A point from the distinguished component of $(\underline{G}/\underline{H})^{L_{0\,\underline{G},\underline{G}/\underline{H}}}$ lies in the distinguished component of $(G/H)^{L_{0\,G,G/H}}$.

Case 3. Here $X = G *_H V$ is an affine homogeneous vector bundle and $\pi : G *_H V \to G/H$ is the natural projection. Applying the algorithm of case 2 to G/H, we compute the lattice $\mathfrak{X}_{G,G/H}$ and find a point x in the distinguished component of $(G/H)^{L_{0\,G,G/H}}$. Applying the following algorithm to the group $L_0 := L_{0\,G,G/H}$ and the L_0-module $V := \pi^{-1}(x)$, we compute $L_{0\,L_0,V}$.

Algorithm 6.0.2. Set $G_0 = L_0, V_0 = V$. Assume that we have already constructed a pair (G_i, V_i), where G_i is an almost connected connected subgroup in G_0 and V_i is a G_i-module. Set $\widetilde{B}_i := B \cap G_i, B_i := \widetilde{B}_i^\circ$. Choose a $\widetilde{B}_i$-semiinvariant vector $\alpha \in V_i^*$. Put $V_{i+1} := (\mathfrak{u}_i^- \alpha)^0$, where $\mathfrak{u}_i^-$ is a maximal unipotent subalgebra of $\mathfrak{g}_i$ normalized by T and opposite to $\mathfrak{b}_i$ and the superscript ^{0} means the annihilator. Put $G_{i+1} := Z_{G_i}(\alpha)$. The group G_{i+1} is almost connected and $L_{0\,G_i,V_i} = L_{0\,G_{i+1},V_{i+1}}$. Note that $\mathrm{rk}[\mathfrak{g}_{i+1}, \mathfrak{g}_{i+1}] \leqslant \mathrm{rk}[\mathfrak{g}_i, \mathfrak{g}_i]$ with the equality iff $\alpha \in V^{[\mathfrak{g}_i, \mathfrak{g}_i]}$. Thus if $[\mathfrak{g}_i, \mathfrak{g}_i]$ acts non-trivially on V, then we may assume that $\mathrm{rk}[\mathfrak{g}_{i+1}, \mathfrak{g}_{i+1}] < \mathrm{rk}[\mathfrak{g}_i, \mathfrak{g}_i]$. So $V_k := V_k^{[\mathfrak{g}_k, \mathfrak{g}_k]}$ for some k. Here $L_{0\,L_0,V} = L_{0\,G_k,V_k}$ coincides with the inefficiency kernel for the action $G_k : V_k$.

By Proposition 3.3.4, $L_{0\,G,X} = L_{0\,L_0,V}$.

Case 4. Suppose $X = G/H$, where H is a nonreductive subgroup of G. We find a parabolic subgroup $Q \subset G$ tamely containing H by using Algorithm 7.1.2 from [Lo5]. Further, we choose a Levi subgroup $M \subset Q$ and $g \in G$ such that $M \cap H$ is a maximal reudctive subgroup of H and gQg^{-1} is an antistandard parabolic subgroup and gMg^{-1} is its standard Levi subgroup. Replace (Q, M, H) with $(gQg^{-1}, gMg^{-1}, gHg^{-1})$. Put $X' := Q^-/H$. Using Remark 3.2.8 from [Lo5], we construct an M-isomorphism of X' with an affine homogeneous vector bundle. By Proposition 3.3.1, $\mathfrak{X}_{G,G/H} = \mathfrak{X}_{M,X'}$. The last lattice is computed as in case 3.

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